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Toric ideals of graphs

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Toric ideals

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{Z}^n$ be a vector configuration in $\mathbb{Q}^n$ and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}_0\}$ the corresponding affine semigroup.
The toric ideal associated with A is the kernel of the K-algebra homomorphism

$$K[x_1, \cdots, x_m] \to K[t_1, \cdots, t_n, t_1^{-1} \cdots t_n^{-1}]$$

given by $\phi(x_i) = t^{\mathbf{a}_i} = t_1^{a_{i,1}} t_2^{a_{i,2}} \cdots t_n^{a_{i,n}}$, where $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \cdots, a_{i,n})$.

Toric ideals

We grade the polynomial ring $K[x_1, \dots, x_m]$ over any field K by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \dots, m$. For $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{N}^m$, we define the *A-degree* of the monomial $\mathbf{x}^\mathbf{u} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_A(\mathbf{x}^\mathbf{u}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

Theorem

The toric ideal I_A *associated to* A *is the prime ideal generated by all the binomials* $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$ *such that* $\deg_A(\mathbf{x}^\mathbf{u}) = \deg_A(\mathbf{x}^\mathbf{v})$.

For such binomials, we define $\deg_A(\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}) := \deg_A(\mathbf{x}^\mathbf{u})$.

Graphs

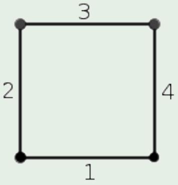
A simple graph G consists of a set of vertices $V(G) = \{v_1, \ldots, v_n\}$ and a set of edges $E(G) = \{e_1, \ldots, e_m\}$, where an edge $e \in E(G)$ is an unordered pair of vertices, $\{v_i, v_j\}$. Let $\mathbb{K}[e_1, \ldots, e_m]$ the polynomial ring in the m variables $e_1, \ldots, e_m$ over a field $\mathbb{K}$. We will associate each edge $e = \{v_i, v_j\} \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$ with basis the set of vertices of G.

With I_G we denote the toric ideal I_{A_G} in $\mathbb{K}[e_1, \ldots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$.

Toric ideals of graphs

Example

Let G be the following graph with 4 vertices and 4 edges.



Then $A_G = \{(1,1,0,0),(0,1,1,0),(0,0,1,1),(1,0,0,1)\}$.
The toric ideal associated with A_G is the kernel of the K-algebra homomorphism

$$K[e_1,e_2,e_3,e_4] \to K[t_1,t_2,t_3,t_4]$$

given by $\phi(e_1) = t_1t_2$, $\phi(e_2) = t_2t_3$, $\phi(e_3) = t_3t_4$, $\phi(e_4) = t_1t_4$.

$$I_G = (e_1e_3 - e_2e_4).$$

Graphs

Definition

- A walk connecting $v_{i_1} \in V(G)$ and $v_{i_{q+1}} \in V(G)$ is a finite sequence of the form
$$w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$$
with each $e_{i_j} = \{v_{i_j}, v_{i_{j+1}}\} \in E(G)$.
- We call a walk $w' = (e_{j_1}, \ldots, e_{j_t})$ a subwalk of w if $e_{j_1} \cdots e_{j_t} | e_{i_1} \cdots e_{i_q}$.
- Length of the walk w is called the number q of edges of the walk.
- An even walk is a walk of even length.
- An odd walk is a walk of odd length.

Graphs

Definition

A walk $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$ is called closed if $v_{i_{q+1}} = v_{i_1}$.
A cycle is a closed walk

$$(\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_1}\})$$

with $v_{i_k} \neq v_{i_j}$, for every $1 \leq k < j \leq q$.

Note that, although the graph G has no multiple edges, the same edge e may appear more than once in a walk. In this case e is called multiple edge of the walk w. If w' is a subwalk of w then it follows from the definition of a subwalk that the multiplicity of an edge in w' is less than or equal to the multiplicity of the same edge in w.

Toric ideals of Graphs

Given an even closed walk

$$w = (e_{i_1}, e_{i_2}, \dots, e_{i_{2q}})$$

of the graph G we denote by

$$E^+(w) = \prod_{k=1}^{q} e_{i_{2k-1}},\; E^-(w) = \prod_{k=1}^{q} e_{i_{2k}}$$

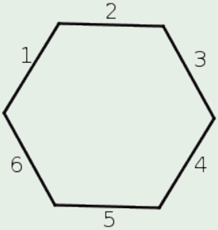
and by B_w the binomial

$$B_w = \prod_{k=1}^{q} e_{i_{2k-1}} - \prod_{k=1}^{q} e_{i_{2k}}$$

belonging to the toric ideal I_G.

Toric ideals of Graphs

Example



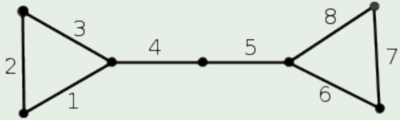
For the even closed walk $w = (e_1, e_2, e_3, e_4, e_5, e_6)$ we have that $E^+(w) = e_1 e_3 e_5$ and $E^-(w) = e_2 e_4 e_6$ therefore

$$B_w = e_1 e_3 e_5 - e_2 e_4 e_6.$$

Note that
$deg_G(e_1 e_3 e_5) = deg_G(e_2 e_4 e_6) = v_1 + v_2 + v_3 + v_4 + v_5 + v_6.$

Toric ideals of Graphs

Example



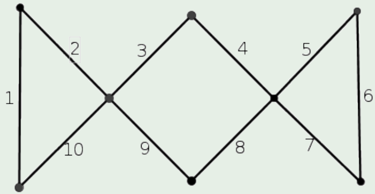
For the even closed walk $w = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_5, e_4)$ we have that $E^+(w) = e_1e_3e_5e_7e_5$ and $E^-(w) = e_2e_4e_6e_8e_4$ therefore

$$B_w = e_1e_3e_5^2e_7 - e_2e_4^2e_6e_8.$$

Note that
$deg_G(e_1e_3e_5^2e_7) = deg_G(e_2e_4^2e_6e_8) = v_1+v_2+2v_3+2v_4+2v_5+v_6+v_7$.

Toric ideals of Graphs

Example

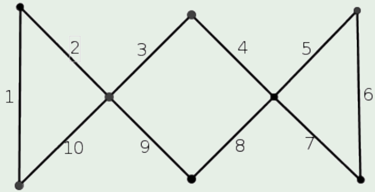


Note that different walks may correspond to the same binomial. For example both walks $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10})$ and $(e_1, e_2, e_9, e_8, e_5, e_6, e_7, e_4, e_3, e_{10})$ correspond to the same binomial

$$B_w = e_1e_3e_5e_7e_9 - e_2e_4e_6e_8e_{10}.$$

Toric ideals of Graphs

Example

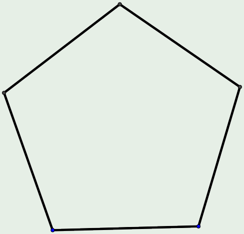


Also note that for certain even closed walks w the binomial B_w may be zero, for example take w to be the even closed walk $(e_1, e_2, e_9, e_8, e_5, e_5, e_8, e_9, e_2, e_1)$ we have

$$B_w = e_1e_9e_5e_8e_2 - e_2e_8e_5e_9e_1 = 0.$$

Toric ideals of Graphs

Example



There are examples that for every even closed walk w the binomial B_w is zero, in these cases

$$I_G = 0.$$

Toric ideals of Graphs

Theorem

(R. Villarreal) The toric ideal I_G of a graph G is generated by binomials of the form B_w, where w is an even closed walk.

Graver basis

Definition

An irreducible binomial $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ in I_A is called *primitive* if there exists no other binomial $x^{\mathbf{v}^+} - x^{\mathbf{v}^-} \in I_A$ such that $x^{\mathbf{v}^+}$ divides $x^{\mathbf{u}^+}$ and $x^{\mathbf{v}^-}$ divides $x^{\mathbf{u}^-}$.

Definition

The set of all primitive binomials of a toric ideal I_A is called the Graver basis of I_A.

Graver basis

The Graver basis is very important.

- Every circuit belongs to the Graver basis
- Every reduced Gröbner basis is a subset of the Graver basis
- The universal Gröbner basis is a subset of the Graver basis
- Every indispensable binomial belongs to the Graver basis
- There are minimal system of generators that are subsets of the Graver basis

Toric ideals of Graphs

Definition

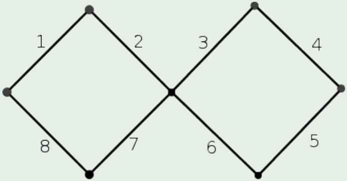
An even closed walk $w = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ is said to be primitive if $B_w \neq 0$ and there exists no even closed subwalk ξ of w of smaller length such that $E^+(\xi)|E^+(w)$ and $E^-(\xi)|E^-(w)$.

Theorem

The walk w is primitive if and only if the binomial B_w is primitive.

Toric ideals of Graphs

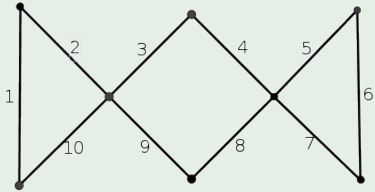
Example



The walk $w = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)$ of the graph G is not primitive, since there exists a closed even subwalk of w, for example $\xi = (e_1, e_2, e_7, e_8)$ such that $e_1 e_7 | e_1 e_3 e_5 e_7$ and $e_2 e_8 | e_2 e_4 e_6 e_8$. Note that $B_w = e_1 e_3 e_5 e_7 - e_2 e_4 e_6 e_8$ and $B_\xi = e_1 e_7 - e_2 e_8$.

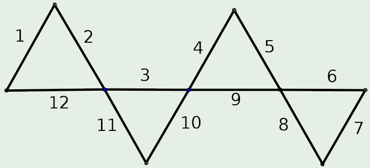
Toric ideals of Graphs

Example



The walk $w = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10})$ in the graph G is primitive, although there exists an even closed subwalk $\xi = (e_3, e_4, e_8, e_9)$, but neither e_3e_8 divides $e_1e_3e_5e_7e_9$ nor e_4e_9 divides $e_1e_3e_5e_7e_9$.
Note that $B_w = e_1e_3e_5e_7e_9 - e_2e_4e_6e_8e_{10}$ and $B_\xi = e_3e_8 - e_4e_9$.

Example



The walk $w = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12})$ is not primitive, since for the walk $\xi = (e_1, e_2, e_3, e_{10}, e_{11}, e_{12})$ we have that ξ is an even closed subwalk of w, $e_1e_3e_{11}|e_1e_3e_5e_7e_9e_{11}$ and $e_2e_{10}e_{12}|e_2e_4e_6e_8e_{10}e_{12}$. Note that $B_w = e_1e_3e_5e_7e_9e_{11} - e_2e_4e_6e_8e_{10}e_{12}$ and $B_\xi = e_1e_3e_{11} - e_2e_{10}e_{12}$.

Toric ideals of Graphs

In a toric ideal of a graph what are elements of the Graver basis?
What are the primitive even closed walks?

Toric ideals of Graphs

In a toric ideal of a graph what are elements of the Graver basis?
What are the primitive even closed walks?

Theorem

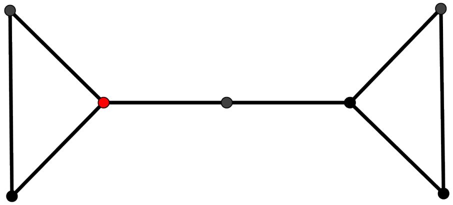
Let G a graph and w an even closed walk of G. The walk w is primitive if and only if

1. *every block of $\mathbf{w}$ is a cycle or a cut edge,*
2. *every multiple edge of the walk w is a double edge of the walk and a cut edge of $\mathbf{w}$,*
3. *every cut vertex of $\mathbf{w}$ belongs to exactly two blocks and it is a sink of both.*

Toric ideals of Graphs

Definition

A cut vertex is a vertex of the graph whose removal increases the number of connected components of the remaining subgraph.



Toric ideals of Graphs

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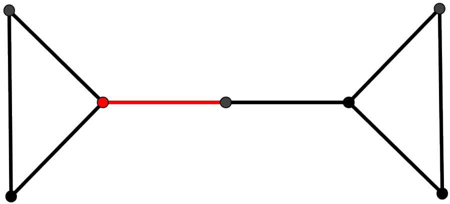
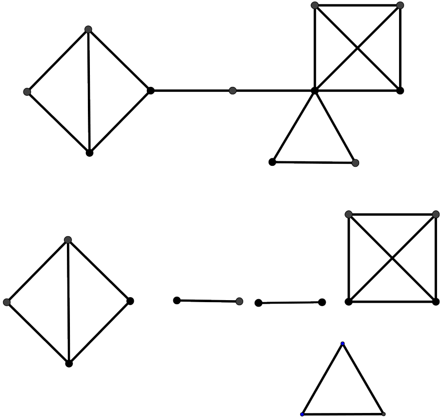


Toric ideals of Graphs

Definition

A graph is called biconnected if it is connected and does not contain a cut vertex.
A block is a maximal biconnected subgraph of a given graph G.



Toric ideals of Graphs

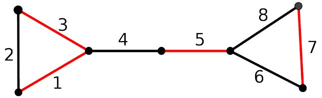
Every even primitive walk $w = (e_{i_1}, \ldots, e_{i_{2k}})$ partitions the set of edges in the two sets $w^+ = \{e_{i_j} | j \ odd\}, w^- = \{e_{i_j} | j \ even\}$, otherwise the binomial B_w is not irreducible.
The edges of w^+ are called odd edges of the walk and those of w^- even edges.
Note that for a closed even walk whether an edge is even or odd depends only on the edge that you start counting from. So it is not important to identify whether an edge is even or odd but to separate the edges in the two disjoint classes.

Toric ideals of Graphs

Definition

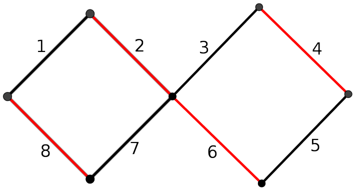
Sink of a block B is a common vertex of two odd or two even edges of the walk w which belong to the block B.

In particular if e is a cut edge of a primitive walk then e appears at least twice in the walk and belongs either to w^+ or w^-. Therefore both vertices of e are sinks.



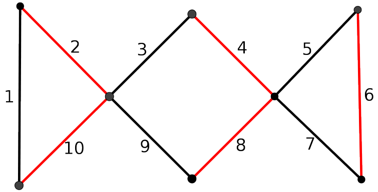
Toric ideals of Graphs

Sink is a property of the walk w and not of the underlying graph w.



For example the walk $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)$ has no sink, while in the walk $(e_1, e_2, e_7, e_8, e_1, e_2, e_7, e_8)$ all four vertices are sinks.

Toric ideals of Graphs



The walk $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10})$ has two cut vertices which are both sinks of all of their blocks.

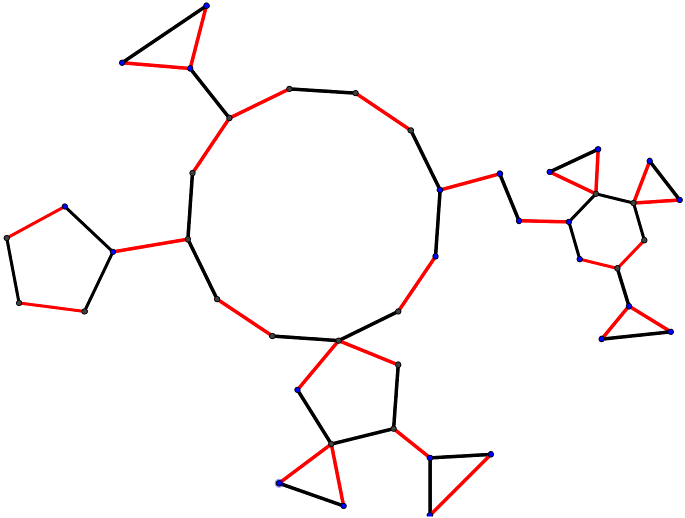
Toric ideals of Graphs

Theorem

Let G a graph and w an even closed walk of G. The walk w is primitive if and only if

1. *every block of $\mathbf{w}$ is a cycle or a cut edge,*
2. *every multiple edge of the walk w is a double edge of the walk and a cut edge of $\mathbf{w}$,*
3. *every cut vertex of $\mathbf{w}$ belongs to exactly two blocks and it is a sink of both.*

Toric ideals of Graphs



Toric ideals of Graphs

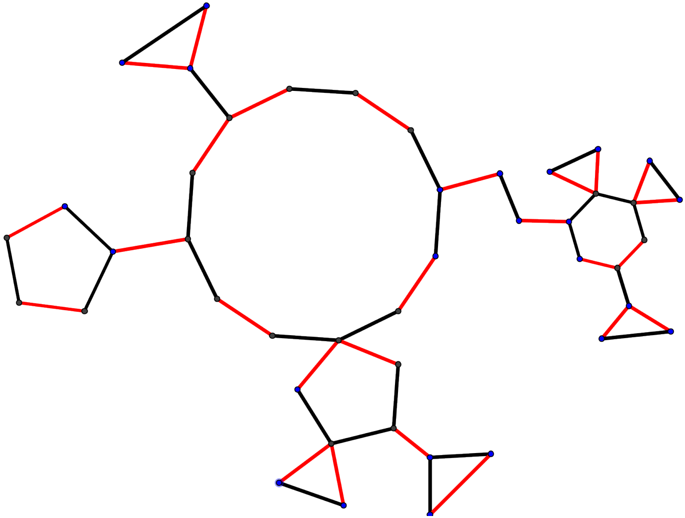
The following theorem describes the underlying graph of a primitive walk.

Theorem

Let G be a graph and let W be a connected subgraph of G. The subgraph W is the graph w of a primitive walk w if and only if

1. *W is an even cycle or*
2. *W is not biconnected and*
    1. *every block of W is a cycle or a cut edge and*
    2. *every cut vertex of W belongs to exactly two blocks and separates the graph in two parts, the total number of edges of the cyclic blocks in each part is odd.*

Toric ideals of graphs



Circuits

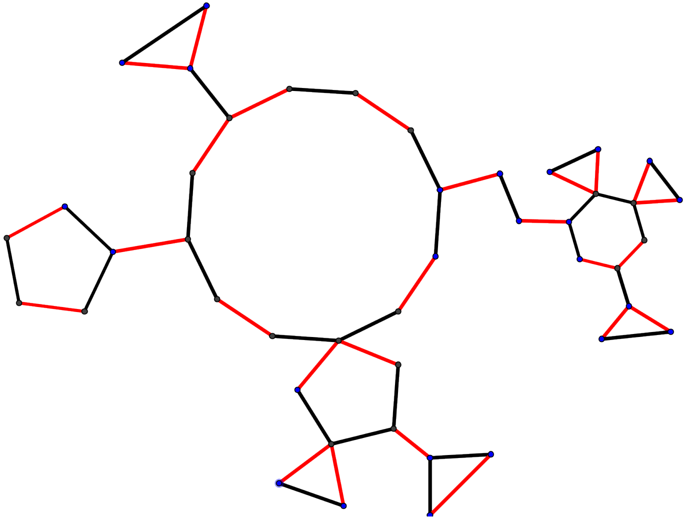
The support of a monomial $x^{\mathbf{u}}$ of $K[x_1, \ldots, x_m]$ is $supp(x^{\mathbf{u}}) := \{i \mid x_i \text{ divides } x^{\mathbf{u}}\}$ and the support of a binomial $B = x^{\mathbf{u}} - x^{\mathbf{v}}$ is $supp(B) := supp(x^{\mathbf{u}}) \cup supp(x^{\mathbf{v}})$. An irreducible binomial B belonging to I_A is called a *circuit* of I_A if there is no binomial $B' \in I_A$ such that $supp(B') \subsetneqq supp(B)$. A binomial $B \in I_A$ is a circuit of I_A if and only if $I_A \cap K[x_i \mid i \in supp(B)]$ is generated by B.

Circuits

Theorem

(B. Sturmfels) The set of circuits of I_A is a subset of both the Universal Gröbner basis and the Graver basis of I_A.

Circuits



Circuits

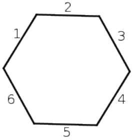
A necessary and sufficient characterization of circuits was given by R. Villarreal:

Theorem

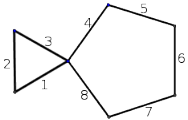
Let G be a finite connected graph. The binomial $B \in I_G$ is circuit if and only if $B = B_w$ where

1. *w is an even cycle or*
2. *two odd cycles intersecting in exactly one vertex or*
3. *two vertex disjoint odd cycles joined by a path.*

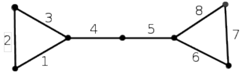
Circuits



w is an even cycle

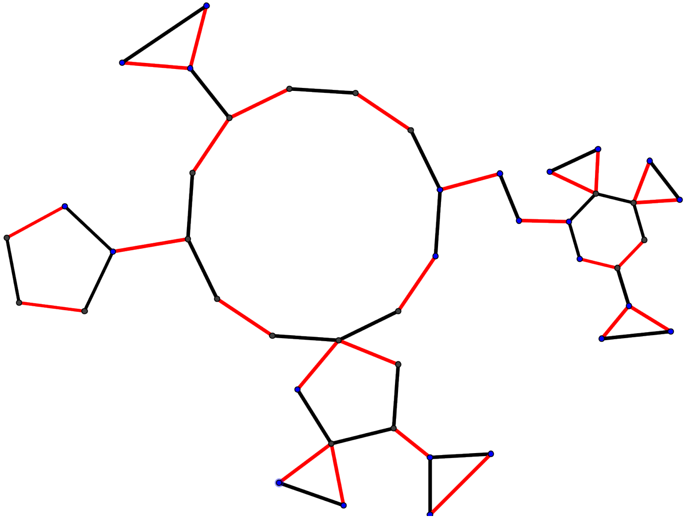


two odd cycles intersecting in exactly one vertex



two vertex disjoint odd cycles joined by a path

Circuits



Toric ideals of Graphs

The knowledge of the form of the circuits, the elements of the Graver basis, the minimal systems of generators and the elements of the universal Gröbner basis of the toric ideal of a graph G, allow us to produce examples of toric ideals having specific properties.

True circuit conjecture

B. Sturmfels in 1995 with the help of S. Hosten and R. Thomas made the following conjecture:

Conjecture

The degree of any element in the Graver basis Gr_A of a toric ideal I_A is bounded above by the maximal true degree of any circuit in $\mathcal{C}_A$.

True circuit conjecture

Consider any circuit C of I_A and regard its support $\mathrm{supp}(C)$ as a subset of A.

Definition

The index of the circuit C, $\mathrm{index}(C)$, is the index of $\mathbb{Z}(\mathrm{supp}(C))$ in $R(\mathrm{supp}(C)) \cap \mathbb{Z}A$.

Definition

The *true degree* of the circuit C is the product $\deg(C) \cdot \mathrm{index}(C)$.

There are several examples of families of toric ideals where circuits do attain the maximum degree. This is also true for families of toric ideals of graphs, for example the binomial that has the maximal degree in I_{K_n} is a circuit.
But this is not true in the general case.

True circuit conjecture