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Universal Gröbner basis and Markov bases of Toric ideals of graphs

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Gröbner bases

By T^n we denote the set of monomials $x^{\mathbf{a}}$ in $k[x_1, \ldots, x_n]$, where $x^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $\mathbf{a} = (a_1, a_2, \cdots, a_n)$.

Definition

By a monomial order on T^n we mean a total order on T^n such that

- $1 < x^{\mathbf{a}}$ for all $x^{\mathbf{a}} \in T^n$ with $x^{\mathbf{a}} \neq 1$
- If $x^{\mathbf{a}} < x^{\mathbf{b}}$ then $x^{\mathbf{a}} x^{\mathbf{c}} < x^{\mathbf{b}} x^{\mathbf{c}}$ for all $x^{\mathbf{c}} \in T^n$.

If $n \geq 2$ then there are infinitely many monomial orders on T^n.

Gröbner bases

Let $<$ be a monomial order on $k[x_1, \ldots, x_n]$. Let f be a nonzero polynomial in $k[x_1, \ldots, x_n]$. We may write

$$f = a_1 x^{\mathbf{u}_1} + a_2 x^{\mathbf{u}_2} + \ldots a_r x^{\mathbf{u}_r},$$

where $a_i \neq 0$ and $x^{\mathbf{u}_1} > x^{\mathbf{u}_2} > \cdots > x^{\mathbf{u}_r}$.

Definition

For $f \neq 0$ in $k[x_1, \ldots, x_n]$, we define the leading monomial of f to be $in_<(f) = x^{\mathbf{u}_1}$. The coefficient a_1 is called the leading coefficient of f and is denoted by $lc(f)$.

Gröbner bases

Definition

A set of non-zero polynomials $G = \{g_1, \dots, g_t\}$ contained in an ideal I, is called Gröbner basis for I if and only if for all nonzero $f \in I$ there exists $i \in \{1, \dots, t\}$ such that $in_<(g_i)$ divides $in_<(f)$.

Definition

A Gröbner basis $G = \{g_1, \dots, g_t\}$ is called a reduced Gröbner basis for I if

- $lc(g_i) = 1$ for all $i \in \{1, \dots, t\}$ and
- no monomial in g_i is divisible by any $in_<(g_j)$ for any $j \neq i$.

Universal Gröbner bases

Although $k[x_1, \dots, x_n]$, for $n \geq 2$ has infinitely many different monomial orders for a fixed nonzero ideal I there exist finitely many different reduced Gröbner bases for I.

Definition

The universal Gröbner basis of an ideal I is the union of all reduced Gröbner bases $G_<$ of the ideal I as $<$ runs over all term orders and is denoted by $UGB(I)$.

The universal Gröbner basis is a finite subset of I and it is a Gröbner basis for I with respect to all term orders simultaneously.

Universal Gröbner bases

The relation among the set of circuits, the Graver basis and the universal Gröbner basis for a toric ideal I_A is given by B. Sturmfels:

Theorem

For any toric ideal I_A we have $Circuits_A \subset UGB_A \subset Graver_A$.

Universal Gröbner bases

For toric ideals of graphs the circuits are described by the following Theorem:

Theorem

(R. Villarreal) Let G be a finite connected graph. The binomial $B \in I_G$ is circuit if and only if $B = B_w$ where

1. *w is an even cycle or*
2. *two odd cycles intersecting in exactly one vertex or*
3. *two vertex-disjoint odd cycles joined by a path.*

Universal Gröbner bases

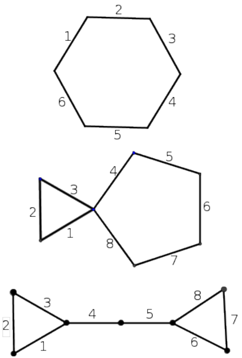
The following theorem describes the underlying graph of a primitive walk.

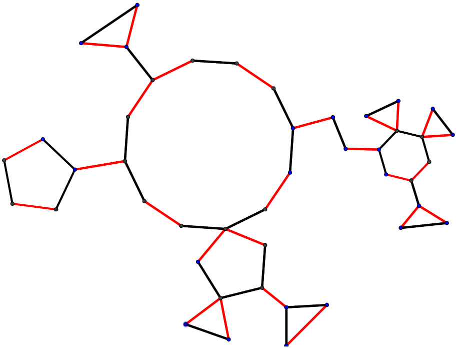
Theorem

Let G be a graph and let W be a connected subgraph of G. The subgraph W is the graph w of a primitive walk w if and only if

1. *W is an even cycle or*
2. *W is not biconnected and*
   1. *every block of W is a cycle or a cut edge and*
   2. *every cut vertex of W belongs to exactly two blocks and separates the graph in two parts, the total number of edges of the cyclic blocks in each part is odd.*

Universal Gröbner bases





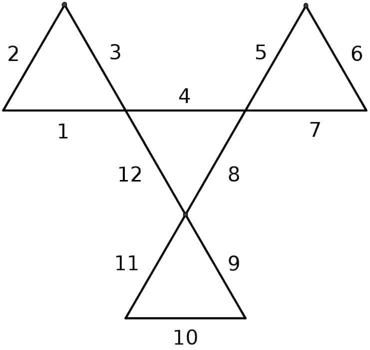
Universal Gröbner bases

J. De Loera, B. Sturmfels and R. Thomas proved that

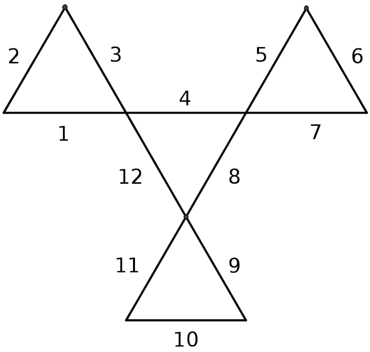
- $\#C_{K_5} = \#UGB_{K_5} = \#Graver_{K_5} = 30$
- $\#C_{K_6} = \#UGB_{K_6} = \#Graver_{K_6} = 285$
- $\#C_{K_7} = \#UGB_{K_7} = \#Graver_{K_7} = 3360$ and
- $\#C_{K_8} = 38010 \neq \#UGB_{K_8} = \#Graver_{K_8} = 45570.$

What is the Universal Gröbner basis of K_n for $n \geq 9$?
What is the Universal Gröbner basis of G for a general graph?

Universal Gröbner bases



Universal Gröbner bases



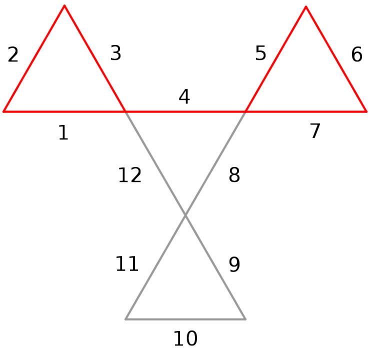
Let w be the walk $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12})$.
We claim that the binomial

$$B_w = e_1e_3e_5e_7e_9e_{11} - e_2e_4e_6e_8e_{10}e_{12}$$

does not belong to the universal Gröbner basis of I_G.
Suppose that there exist a monomial order $<$ such that B_w belongs to the reduced Gröbner basis of I_G with respect to $<$.
There are two cases:

- $e_1e_3e_5e_7e_9e_{11} > e_2e_4e_6e_8e_{10}e_{12}$
- $e_1e_3e_5e_7e_9e_{11} < e_2e_4e_6e_8e_{10}e_{12}$

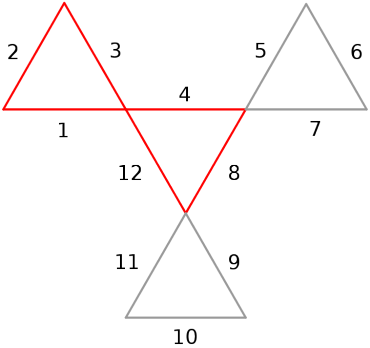
Universal Gröbner bases



First case:
$e_1e_3e_5e_7e_9e_{11} > e_2e_4e_6e_8e_{10}e_{12}$
Look at the binomials of I_G.
$B_1 = e_1e_3e_5e_7 - e_2e_4^2e_6$,
$B_2 = e_5e_7e_9e_{11} - e_6e_8^2e_{10}$,
$B_3 = e_9e_{11}e_1e_3 - e_{10}e_{12}^2e_2$.
Note that $e_1e_3e_5e_7 | e_1e_3e_5e_7e_9e_{11}$, $e_5e_7e_9e_{11} | e_1e_3e_5e_7e_9e_{11}$, and $e_9e_{11}e_1e_3 | e_1e_3e_5e_7e_9e_{11}$.
Therefore $e_1e_3e_5e_7 < e_2e_4^2e_6$, $e_5e_7e_9e_{11} < e_6e_8^2e_{10}$, $e_9e_{11}e_1e_3 < e_{10}e_{12}^2e_2$.

But then $(e_1e_3e_5e_7e_9e_{11})^2 < (e_2e_4e_6e_8e_{10}e_{12})^2$ contradicting $e_1e_3e_5e_7e_9e_{11} > e_2e_4e_6e_8e_{10}e_{12}$.

Universal Gröbner bases

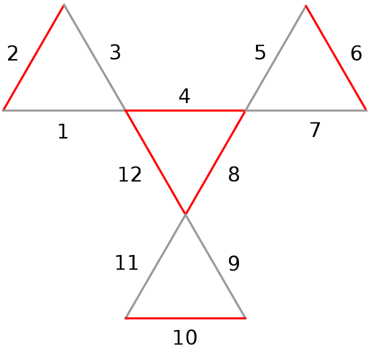


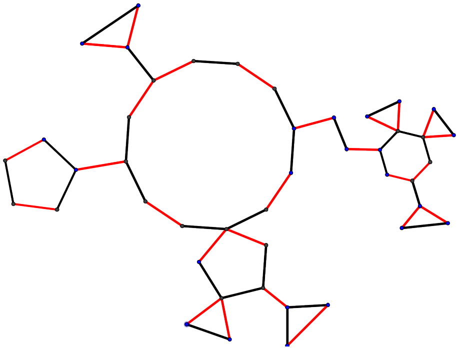
Second case:
$e_1e_3e_5e_7e_9e_{11} < e_2e_4e_6e_8e_{10}e_{12}$
Look at the binomials of I_G.
$G_1 = e_1e_3e_8 - e_2e_4e_{12}$,
$G_2 = e_5e_7e_{12} - e_6e_8e_4$,
$G_3 = e_9e_{11}e_4 - e_{10}e_{12}e_8$.
Note that $e_2e_4e_{12}|e_2e_4e_6e_8e_{10}e_{12}$, $e_6e_8e_4|e_2e_4e_6e_8e_{10}e_{12}$, and $e_{10}e_{12}e_8|e_2e_4e_6e_8e_{10}e_{12}$.
Therefore $e_1e_3e_8 > e_2e_4e_{12}$, $e_5e_7e_{12} > e_6e_8e_4$, $e_9e_{11}e_4 > e_{10}e_{12}e_8$.

But then $(e_4e_8e_{12})(e_1e_3e_5e_7e_9e_{11}) > (e_4e_8e_{12})(e_2e_4e_6e_8e_{10}e_{12})$ contradicting $e_1e_3e_5e_7e_9e_{11} < e_2e_4e_6e_8e_{10}e_{12}$.

Universal Gröbner bases

The existence of this walk implies for $n \geq 9$ that $UGB_{K_n} \neq Gr_{K_n}$, where K_n is the complete graph on n vertices.





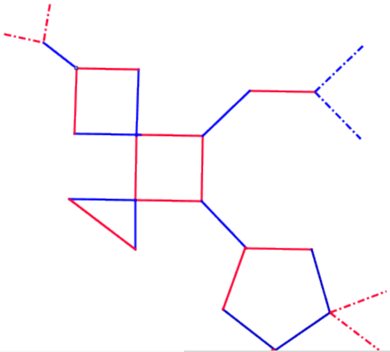
Universal Gröbner bases

Definition

A cyclic block B of a primitive walk w is called pure if all edges of B are either in $E^+(w)$ or in $E^-(w)$.

Theorem

Let w be an even primitive walk that has a pure cyclic block. Then B_w does not belong to the universal Gröbner basis of I_G.



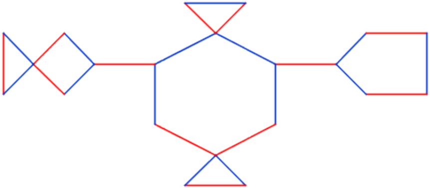
Universal Gröbner bases

Definition

A primitive walk w is called mixed if no cyclic block of w is pure.

Theorem

Let w be a primitive walk. B_w belongs to the universal Gröbner basis of I_G if and only if w is mixed.



Universal Gröbner bases

Theorem

Let w be a primitive walk. B_w belongs to the universal Gröbner basis of I_G if and only if w is mixed.

Sketch of the proof. For any mixed primitive walk w we construct a term order $<_w$ that depends on w to prove that B_w belongs to the reduced Gröbner basis with respect to $<_w$.
It is enough to prove that whenever there exists a primitive binomial B_z such that $E^+(z)|E^+(w)$ then $E^-(z) >_w E^+(z)$. Note that $E^-(z) \nmid E^-(w)$ since w is primitive and $E^-(z) \nmid E^+(w)$ since w is mixed.

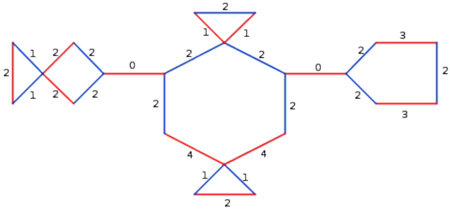
Universal Gröbner bases

Let w be a mixed primitive walk. We define a term order $<_w$ on $\mathbb{K}[e_1, \ldots, e_n]$, as an elimination order with the variables that do not belong to $\mathbf{w}$ larger than the variables in $\mathbf{w}$. We order the first set of variables by any term order, and the second set of variables as follows: Let $B_1, \ldots B_{s_0}$ be any enumeration of all cyclic blocks of $\mathbf{w}$. Let t_i^+ denotes the number of edges in $\mathbf{w}^+ \cap B_i$ and t_i^- denotes the number of edges in $\mathbf{w}^- \cap B_i$. Let $W = (w_{ij})$ be the $(s_0) \times m$ matrix

$$w_{ij} = \begin{cases} 0, & \text{if } e_j \notin B_i, \\ t_i^-, & \text{if } e_j \in B_i \cap \mathbf{w}^+, \\ t_i^+, & \text{if } e_j \in B_i \cap \mathbf{w}^- \end{cases}$$

where m is the number of edges of $\mathbf{w}$.
Denote by $[u]$ the vector u written as a column vector. We say that $e^u <_w e^v$ if and only if the first nonzero coordinate of $W[u-v]$ is negative, otherwise, if $W[u-v] = \mathbf{0}$, order them lexicographically.

Universal Gröbner bases



Universal Gröbner bases

Let w be a mixed primitive walk. We define a term order $<_w$ on $\mathbb{K}[e_1, \ldots, e_n]$, as an elimination order with the variables that do not belong to $\mathbf{w}$ larger than the variables in $\mathbf{w}$. We order the first set of variables by any term order, and the second set of variables as follows: Let $B_1, \ldots B_{s_0}$ be any enumeration of all cyclic blocks of $\mathbf{w}$. Let t_i^+ denotes the number of edges in $\mathbf{w}^+ \cap B_i$ and t_i^- denotes the number of edges in $\mathbf{w}^- \cap B_i$. Let $W = (w_{ij})$ be the $(s_0) \times m$ matrix

$$w_{ij} = \begin{cases} 0, & \text{if } e_j \notin B_i, \\ t_i^-, & \text{if } e_j \in B_i \cap \mathbf{w}^+, \\ t_i^+, & \text{if } e_j \in B_i \cap \mathbf{w}^- \end{cases}$$

where m is the number of edges of $\mathbf{w}$.
Denote by $[u]$ the vector u written as a column vector. We say that $e^u <_w e^v$ if and only if the first nonzero coordinate of $W[u-v]$ is negative, otherwise, if $W[u-v] = \mathbf{0}$, order them lexicographically.

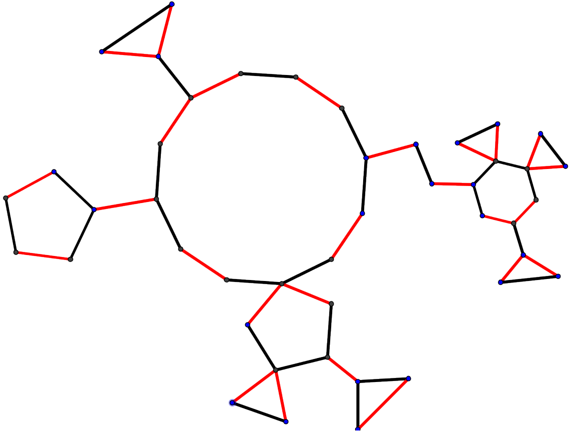
Universal Markov basis

The aim is to characterize the walks w of the graph G such that the binomial B_w belongs to a minimal system of generators (a markov basis) of the ideal I_G.
Certainly the walk has to be primitive, but this is not enough.
The walk must have more properties, the first one depends on the graph w and the rest on the induced graph G_w of w.

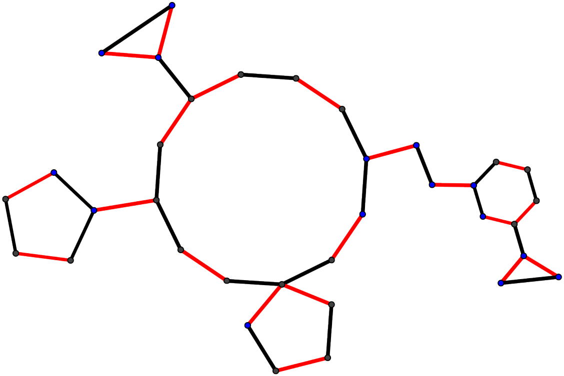
Universal Markov basis

Definition

We call strongly primitive walk a primitive walk that has not two cut points with distance one in any cyclic block.



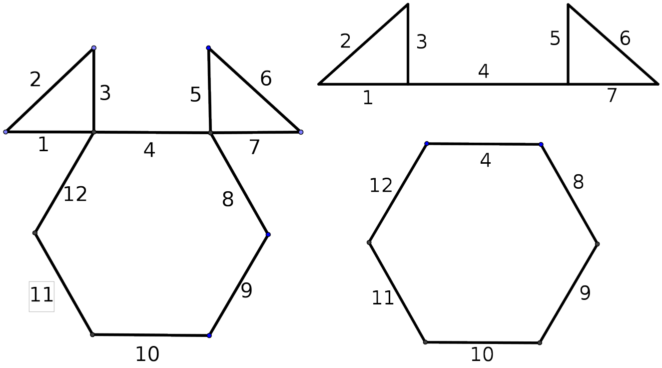
Universal Markov basis



Theorem

Let w be an even closed walk such that the binomial B_w is minimal then the walk w is strongly primitive.

Universal Markov basis



$$B_w = e_1 e_3 e_5 e_7 e_9 e_{11} - e_2 e_4 e_6 e_8 e_{10} e_{12}$$

is not minimal since

$$B_w = (e_1 e_3 e_5 e_7 - e_2 e_4^2 e_6) e_9 e_{11} - (e_4 e_9 e_{11} - e_8 e_{10} e_{12}) e_2 e_4 e_6.$$

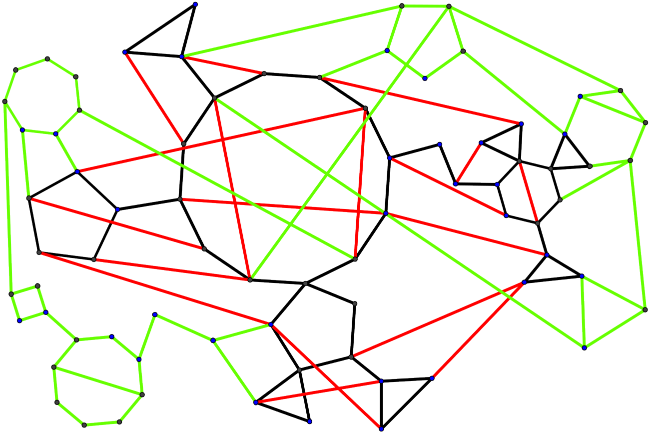
Universal Markov basis

While the property of a walk to be primitive depends only on the graph $\mathbf{w}$, the property of the walk to be minimal or indispensable depends also on the induced graph G_w.

Definition

If W is a subset of the vertex set $V(G)$ of G then the *induced subgraph* of G on W is the subgraph of G whose vertex set is W and whose edge set is $\{\{v, u\} \in E(G) | v, u \in W\}$. When w is a closed walk we denote by G_w the induced graph of G on the set of vertices $V(\mathbf{w})$ of $\mathbf{w}$.

Induced subraph



Universal Markov basis

An edge f of the graph G is called a *chord* of the walk w if the vertices of the edge f belong to $V(\mathbf{w})$ and $f \notin E(\mathbf{w})$.
In other words an edge is called chord of the walk w if it belongs to $E(G_w)$ but not in $E(\mathbf{w})$.
Let w be an even closed walk $((v_1, v_2), (v_2, v_3), \ldots, (v_{2k}, v_1))$ and $f = \{v_i, v_j\}$ a chord of w. Then f breaks w in two walks:

$$w_1 = (e_1, \ldots, e_{i-1}, f, e_j, \ldots, e_{2k})$$

and

$$w_2 = (e_i, \ldots, e_{j-1}, f),$$

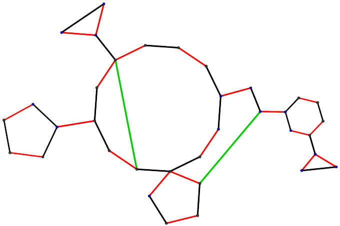
where $e_s = (v_s, v_{s+1}),\ 1 \leq s \leq 2k - 1$ and $e_{2k} = (v_{2k}, v_1)$. The two walks are both even or both odd.

Universal Markov basis

We partition the set of chords of a primitive even walk in three parts: bridges, even chords and odd chords.

Definition

A chord $f = \{v_1, v_2\}$ is called bridge of a primitive walk w if there exist two different blocks B_1, B_2 of w such that $v_1 \in B_1$ and $v_2 \in B_2$.



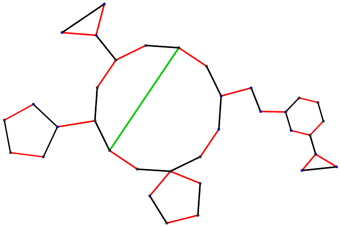
Theorem

Let w be a primitive walk. If B_w is a minimal binomial then w has no bridge.

Universal Markov basis

Definition

A chord is called even if it is not a bridge and breaks the walk in two even walks.
A chord is called odd if it is not a bridge and breaks the walk in two odd walks.

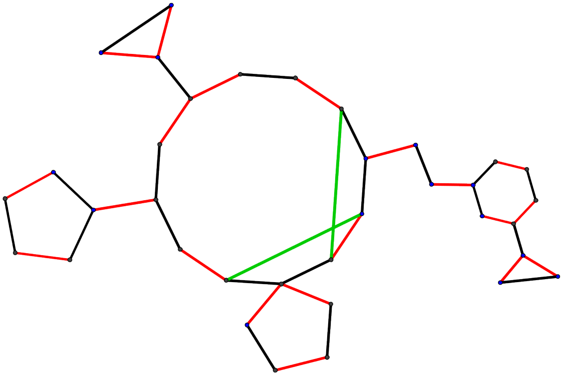


Theorem

Let w be a primitive walk. If B_w is a minimal binomial then w has no even chord.

Definition

Let $w = ((v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{2q}}, v_{i_1}))$ be a primitive walk. Let $f = \{v_{i_s}, v_{i_j}\}$ and $f' = \{v_{i_{s'}}, v_{i_{j'}}\}$ be two odd chords (that means not bridges and $j - s, j' - s'$ are even) with $1 \leq s < j \leq 2q$ and $1 \leq s' < j' \leq 2q$. We say that f and f' cross effectively in w if $s' - s$ is odd (then necessarily $j - s', j' - j, j' - s$ are odd) and either $s < s' < j < j'$ or $s' < s < j' < j$.

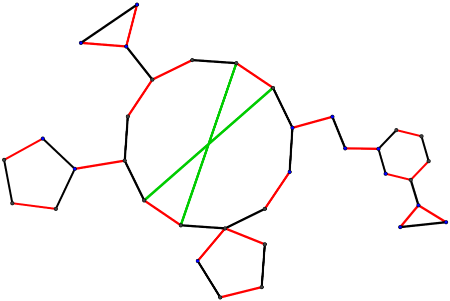


Universal Markov basis

Note that if two odd chords f and f' cross effectively in w then all of their vertices are in the same cyclic block of w.

Definition

We call an F_4 of the walk w a cycle (e, f, e', f') of length four which consists of two edges e, e' of the walk w either both odd or both even, and two odd chords f and f' which cross effectively in w.



Universal Markov basis

Definition

Let w be a primitive walk and f, f' be two odd chords. We say that f, f' cross strongly effectively in w if they cross effectively and they do not form an F_4 in w.

Theorem

Let w be a primitive walk. If B_w is a minimal binomial then all the chords of w are odd and there are not two of them which cross strongly effectively.

Toric ideals of Graphs

An F_4, (e_1, f_1, e_2, f_2), separates the vertices of w in two parts $V(\mathbf{w}_1), V(\mathbf{w}_2)$, since both edges e_1, e_2 of the F_4 belong to the same block of $w = (w_1, e_1, w_2, e_2)$.

Definition

We say that an odd chord f of a primitive walk $w = (w_1, e_1, w_2, e_2)$ crosses an F_4, (e_1, f_1, e_2, f_2), if one of the vertices of f is in $V(\mathbf{w}_1)$, the other in $V(\mathbf{w}_2)$ and f is different from f_1, f_2.

Theorem

Let w be a primitive walk. If B_w is a minimal binomial, then no odd chord crosses an F_4 of the walk w.

Toric ideals of Graphs

Theorem

Let w be an even closed walk. B_w belongs to the universal Markov basis if and only if

1. *w is strongly primitive,*
2. *all the chords of w are odd and there are not two of them which cross strongly effectively and*
3. *no odd chord crosses an F_4 of the walk w.*

Indispensable binomials

Theorem

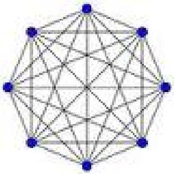
Let w be an even closed walk. B_w is an indispensable binomial if and only if w is a strongly primitive walk, all the chords of w are odd and there are not two of them which cross effectively.

We have that if B_w is indispensable then w has no F_4 and if B_w is minimal but not indispensable then w has at least one F_4. If no minimal generator has an F_4 then the toric ideal is generated by indispensable binomials, so the ideal I_G has a unique system of binomial generators and conversely.

Theorem

Let G be a graph which has no cycles of length four. The toric ideal I_G has a unique system of binomial generators.

Graver basis



Toric ideals of Graphs

- E. Reyes, Ch. Tatakis, ——-, *Minimal generators of toric ideals of graphs,* Advances in Applied Mathematics, 48 (2012) 64-78.
- Ch. Tatakis, ——–, *On the universal Gröbner of toric ideals of graphs,* Journal of Combinatorial Theory, Series A, 118 (2011) 1540-1548.