PROBLEM: A Γ-EQUIVARIANT SETTING FOR MATROIDS

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The problem: Let Γ be a group. How can a group action be defined on a matroid and a Γ-equivariant morphismn in the corresponding category.

We begin with a quick review of the definition of a matroid, weak maps and describe some representations of diverse matroids.

Definition. A *matroid* M is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I} \subseteq 2^E$ satisfying

- $\emptyset \in \mathcal{I}$;
- if $Y \in \mathcal{I}$ and $X \subseteq Y$, then $X \in \mathcal{I}$;
- if $X, Y \in \mathcal{I}$ with $|X| < |Y|$, there exists $y \in Y - X$ such that $X \cup \{y\} \in \mathcal{I}$.

The set $E = E(M)$ is called the *ground set* of M, the elements of E are the *atoms* and $\mathcal{I} = \mathcal{I}(M)$ is called the collection of *independent sets* of M.

A subset of the ground set E that is not independent is called *dependent*and the collection denoted by $\mathcal{D}(M)$. A maximal independent set with respect to inclusion is called a *basis* for the matroid. A *circuit* in a matroid M is a minimal dependent subset of E, that is a dependent set whose proper subsets are all independent and are denoted by $\mathcal{C}(M)$. The terminology comes from graph theory as we will see below.

There are also two functions on the powerset of E:

The *rank function* $rk_M : 2^E \to \mathbb{N}$ is given by

$$rk_M(X) = \max\{|Y| : Y \subseteq X \text{and} Y \in \mathcal{I}\}$$

We define the *rank* of M, denoted by $rk(M)$, by $rk_M(E)$ and it is obviously the same as the cardinality of any basis.

The *closure* of a set $X \subseteq E$ is

$$cl(X) = \{x \in E : rk(X \cup \{x\}) = rk(X)\}.$$

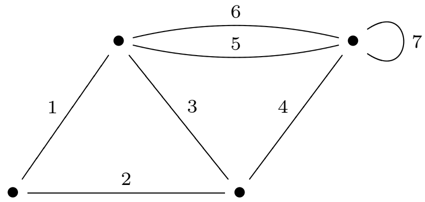
A *flat* of M is any set $X \subseteq E$ where $X = cl(X)$. Flats of rank $rk(M) - 1$ are also called *hyperplanes* or *coatoms* and are denoted by $\mathcal{T}(M)$.

The dependent sets, the bases, the circuits, the rank funktion, the flats, the hyperplanes of a matroid each of them characterize the matroid completely and furthermore each of these sets may also be taken as axioms for a matroid.

Matroid theory developed mainly out of the properties of independence and dimension in vector spaces. If E is any finite subset of a vector space V, then we can define a matroid M on E by taking the independent sets of M to be the *linearly independent* subsets of E, these are the so called *vector matroids*. A matroid that is equivalent to a vector matroid, although it may be presented differently, is called *representable*.

For our example we use another source for the theory of matroids: graph theory. Take as E the set of all edges in a finite graph G and consider a set of edges independent if and only if it does not contain a simple cycle. This is called the *graphic matroid* of G.

Example 1. Our graph G has 4 vertices and 7 edges, labeled by numbers:



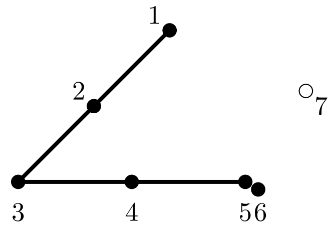
Therefore $E = [7]$ and the Matroid M_G is defined by the following independent set:

$$\mathcal{I}(M_G) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \{2,4,5\}, \{2,4,6\}\}$$

This set is quite huge however the set of hyperplanes is manageable:

$$\mathcal{T}(M_G) = \{\{1,2,3,7\}, \{3,4,5,6,7\}, \{2,4,7\}, \{1,4,7\}, \{2,5,6,7\}, \{1,5,6,7\}\}$$

One attractive feature of graphic matroids is that one can determine many properties of such matroids from the picture of the graphs, for example the minimal circles correspond to circuits. For matroids of small rank we have a geometric representation that is similarly useful. For example for $rk(M) = 3$ the atoms are presented by labeled points and they are linearly dependent then they are collinear, rank zero atoms correspond to an empty bullet and two linear dependent atoms are presented by two points cuddled up to each other. Therefore it is easy to read of the defining hyperplanes from this picture. The example 1 has the following geometric representation:



Next, we introduce a structure-preserving map between matroids.

Definition. A *weak map* is a function $\tau : M \to N$ satisfying the condition if $X \subseteq E(M)$ such that $\tau|_X$ is injective and $\tau(X) \in \mathcal{I}(N)$, then $X \in \mathcal{I}(M)$.

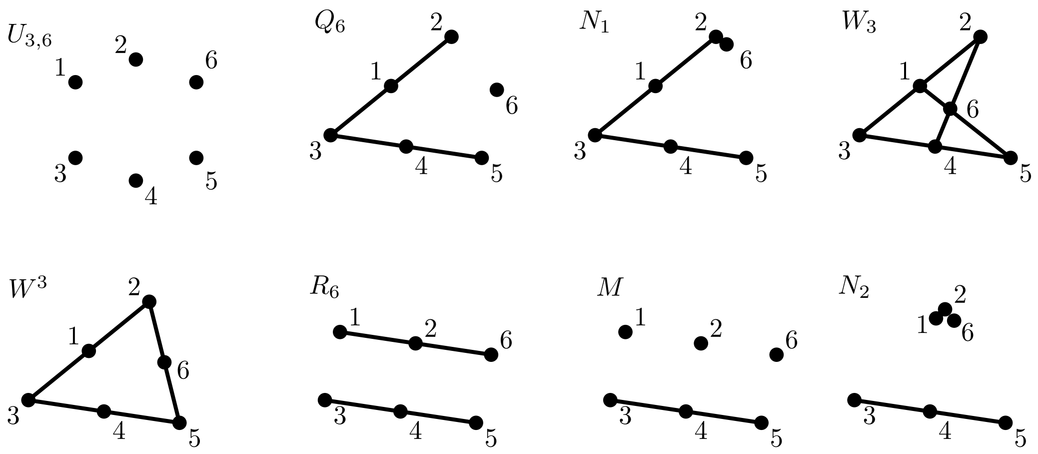
If $\tau : M \to N$ is a surjactive weak map, then each $Y \in E(N)$ can be rewritten as $Y' = \tau^{-1}(Y)$ to form a matroid $N' \cong N$ with ground set $E(M)$ such that τ is equivalent to $id : M \to N'$. So it is sufficient here to restrict our attention to identity maps between matroids on the same ground set and with that restriction the following Proposition is very useful:

Proposition. Let M and N be matroids on a common ground set E. The following are equivalent:

(i) The identity map on E is a weak map from M to N.
(ii) Every dependent set in M is dependent in N.
(iii) Every circuit of M contains a circle of N.
(iv) For every subset X of E: $rk_M(X) \geq rk_N(X)$.

The collection $\mathcal{E}$ of matroids on a fixed ground set E can be partially ordered by taking $M_1 \geq M_2$ if the identity map on E is a weak map from M_1 to M_2, it is called the *weak order* on $\mathcal{E}$. The maximal element in this setting is the free matroid on E, with every subset being independent, and the minimal one is the rank-0 matroid, with every atom being dependent. Fixing the rank to r we write for the corresponding collection $\mathcal{E}_r$ and the unique maximal element with respect to the weak order is the uniform matroid $U_{r,|E|}$, with all subsets of cardinality r forming a basis, but there is no unique minimal Element.

Example 2. Let $E = [6]$. We have the following matroids on E of rank three.



We can illustrate their weak partial ordering in a Hasse-diagram:

$$
\begin{tikzcd}
U_{3,6} \arrow[dr] & & \\
& M \arrow[dl] \arrow[dr] & \\
Q_6 \arrow[dr] \arrow[dd] & & R_6 \arrow[dd] \\
& W^3 \arrow[dr] & \\
W_3 \arrow[dr] & & N_1 \arrow[dl] \\
& N_2 &
\end{tikzcd}
$$

An action of a group Γ on a matroid should preserve the defining structure of this matroid.

Definition. Let Γ be a group and M a matroid with ground set E. A *group action* of Γ on M is a binary operator:

$$\circ : \Gamma \times E \to E$$

that satisfies the following axioms:

(i) $(\pi \cdot \sigma) \circ x = \pi \circ (\sigma \circ x) \quad \forall \pi, \sigma \in \Gamma \quad \text{and} \quad x \in E,$
(ii) $id \circ x = x \quad \forall x \in E$ and id the identity element of Γ,
(iii) $\forall X \in \mathcal{D}(M) : \quad \pi \circ X := \{\pi \circ x : x \in X\} \in \mathcal{D}(M)$
or in short: $\pi \circ$ dependent = dependent.

A group action is said to be *faithful*, if $\forall \pi, \sigma \in \Gamma$ with $\pi \neq \sigma$ $\exists x \in E : \quad \pi \circ x \neq \sigma \circ x$.

We are only interested in faithful group actions, therefore we can identify Γ with a subgroup of the symmetric group $S_{|E|}$. Due to the fact that Γ has finite order the condition (iii) is equivalent to: $\pi \circ$ independent = independent.

This definition of a faithful group action actually agrees with the usual definition of an automorphism of matroids, see Welsh. Hence the automorphism group $Aut(M)$ is the maximal group acting on a matroid and every other group acting on a matroid is isomorphic to a subgroup of $Aut(M)$. For graphical matroids $Aut(M)$ is isomorphic to the cycle automorphism group $A_C(G)$ of a corresponding graph G, where $A_C(G)$ is the collection of permutations $\pi : E(G) \to E(G)$ such that $X \subseteq E(G)$ is a cycle iff $\pi(X) \subseteq E(G)$ is a cycle.

Proposition (Welsh)**.** Given any finite group H there exists a matroid M with $Aut(M) \cong H$.

A detailed study of the properties of the automorphism group was made by Piff.

Let's have a look at the automorphism goups of the matroids presented in example 2. As all the possible defining sets of a matroid are invariant under $Aut(M)$ we usually choose the one with smallest cardinality.

- The uniform matroid $U_{3,6}$: $Aut(U_{3,6}) = S_6$.
  Moreover, if $E(M) = [n]$: $\quad Aut(M) = S_n \quad \Leftrightarrow \quad M = U_{r,n}$ for some r.
- M with $\mathcal{C}(M) = \{\{3,4,5\}\} \quad \Rightarrow \quad Aut(M) = \langle (34),(45),(12),(16)\rangle \cong S_3 \times S_3$.
- Q_6 With $\mathcal{C}(Q_6) = \{\{1,2,3\},\{3,4,5\}\} \;\Rightarrow\; Aut(Q_6) = \langle (12),(1524)\rangle \cong D_4$ the dihedral group of order 8.
- R_6 with $\mathcal{C}(R_6) = \{\{1,2,6\},\{3,4,5\}\} \;\Rightarrow\; Aut(R_6) = \langle (34),(45),(12),(16)\rangle \cong S_3 \times S_3$.
- W^3 with $\mathcal{C}(W^3) = \{\{1,2,3\},\{3,4,5\},\{1,5,6\}\} \;\Rightarrow\; Aut(W^3) = \langle (24)(15),(13)(46)\rangle \cong D_3$ the symmetry group of the triangle.
- N_1 with $\mathcal{T}(N_1) = \{\{3,4,5\},\{1,2,3,6\},\{2,4\},\{2,5\},\{1,4,6\},\{1,5,6\}\}$
  $\Rightarrow\; Aut(N_1) = \langle (45),(16)\rangle \cong S_2 \times S_2$ the Klein group.
- $W_3 \cong M(K_4)$ the matroid corresponding to the complete graph on four vertices with $\mathcal{T}(W_3) = \{\{1,2,3\},\{3,4,5\},\{1,4,6\},\{2,5,6\},\{3,6\},\{2,4\},\{1,5\}\}$
  $\Rightarrow\; Aut(W_3) = \langle (45)(12),(36)(24)\rangle \cong D_4$.
- N_2 with $\mathcal{C}(N_2) = \{\{3,4,5\},\{1,2\},\{1,6\},\{2,6\}\}$
  $\Rightarrow\; Aut(N_2) = \langle (34),(45),(12),(16)\rangle \cong S_3 \times S_3$.

Having a nice lattice for this collection of matroids we hoped to get a similar lattice for the corresponding automorphism groups by inclusion.

$$S_6$$

$$\langle (34),(45),(12),(16)\rangle \cong S_3 \times S_3$$

$$\langle (12),(1524)\rangle \cong D_4 \qquad\qquad \langle (34),(45),(12),(16)\rangle \cong S_3 \times S_3$$

$$\langle (24)(15),(13)(46)\rangle \cong D_3$$

$$\langle (45)(12),(36)(24)\rangle \cong D_4 \qquad\qquad \langle (45),(16)\rangle \cong S_2 \times S_2$$

$$\langle (34),(45),(12),(16)\rangle \cong S_3 \times S_3$$

But this obviously did not work out :-(

Definition. Let Γ be acting on the matroids M and N both on the ground set E. A weak map $f : M \to N$ of matroids is called Γ-*equivariant* if $\sigma \circ f(x) = f(\sigma \circ x)\ \forall \sigma \in \Gamma$ and $x \in E$.

Back to example 2:

- The identity maps $id : M \to N_2$ and $id : R_6 \to N_2$ are obviously equivariant under the $S_3 \times S_3$ action.
- Let $S_2 \times S_2 = \{(1), \sigma_1, \sigma_2, \sigma_3\} \cong \{(1), (12), (45), (12)(45)\}$ acting on R_6 and $S_2 \times S_2 = \{(1), \sigma_1, \sigma_2, \sigma_3\} \cong \{(1), (16), (45), (16)(45)\}$ acting on N_1. Now the identity map $id : R_6 \to N_1$ is not $S_2 \times S_2$-equivariant since $\sigma_1 \circ id(1) = \sigma_1 \circ 1 = 6$ but $id(\sigma_1 \circ 1) = id(2) = 2$. However, if we take the permutation $(26) : R_6 \to N_1$ then we get a weak and $S_2 \times S_2$-equivariant map.
- The map $id : Q_6 \to W_3$ is not D_4-equivariant and there exists no surjective weak D_4-equivariant map from Q_6 to W_3. That is because there are only two elements of order four in D_4 and the corresponding permutations on Q_6 have both two fixpoints, however the corresponding permutations of order four on W_3 are fixpointfree. And by $\sigma \circ (f(x)) = f(\sigma \circ x) = f(x)$ for x fixpoint of σ we get a contradiction.