Markov Bases of Lattice Ideals

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Lattice Ideals

Let \( k \) be a field and \( R = k[x_1, \ldots, x_n] \). For \( u = (u_1, \ldots, u_n) \in \mathbb{N}^n \) we let \( x^u = x_1^{u_1} \cdot \cdots \cdot x_n^{u_n} \). Let \( L \subset \mathbb{Z}^n \) be a lattice.

**Definition**

The lattice ideal \( I_L \) is

\[
I_L := \langle x^u - x^v : u - v \in L \rangle = \langle x^{w^+} - x^{w^-} : w \in L \rangle
\]

The papers mentioned on this lecture are
Example: lattice ideals

Let \( L \subset \mathbb{Z}^2 \) be the lattice generated by \((1, -2)\). \( R = \mathbb{k}[x, y] \)

\[
(1, -2) = (1, 0) - (0, 2), \quad I_L = \langle x - y^2 \rangle
\]

The exponents of the binomials in \( I_L \) are in the first quadrant and determine lines parallel to \( L \).

\( I_L \) has a unique minimal binomial generating set.

If \( L \cap \mathbb{N}^m = \{0\} \), we say that \( L \) is positive.
Example: lattice ideals

Let $L = 3\mathbb{Z}$, so $L$ is generated by 3. We work in $R = \mathbb{k}[x]$. 

$u - v \in L$ iff $u \equiv v \mod 3$.

$I_L = \langle 1 - x^3 \rangle$.

But also $I_L = \langle 1 - x^6, x^5 - x^8 \rangle$

$I_L$ has (at least) two minimal generating sets of different cardinality.
A set $S$ is a Markov basis for $I_L$ if $S$ consists of binomials and $S$ is a minimal generating set of $I_L$ of minimal cardinality.

For counting purposes, a binomial $B$ is the same as $-B$. How many “different” Markov bases are there? Can we compute the cardinality of a Markov basis? Are there indispensable binomials or indispensable monomial terms of $I_L$? Is there a characteristic shared by different Markov sets?
Motivation-History


Applications of Lattice Ideals in Algebraic Statistics, Integer Programming, Graph Theory, Hypergeometric Differential Equations, etc.

Most of the applications deal with the cases of positively graded lattices.
Degrees and fibers

$L \subset \mathbb{Z}^m$. Let $a_i = e_i + L$ for $i = 1, \ldots, m$, $A_L = \{a_1, \ldots, a_m\}$.

Recall the $A$-degree of a monomial

$$\text{deg}_{A_L}(x_1^{u_1} \cdots x_m^{u_m}) := u_1a_1 + \cdots + u_ma_m.$$ 

and the property

$$l_l = \langle x^u - x^v : \text{deg}_{A_L}(x^u) = \text{deg}_{A_L}(x^v) \rangle.$$ 

Yesterday we talked about fibers corresponding to vectors of $L$ ($\mathcal{F}_u$ or $\mathcal{F}_{u^+}$). We recall the definition and emphasize the connection to the $A$-degree.

**Definition**

Let $b \in \mathbb{N}A_L$. The fiber at $b$ is the following set of monomials:

$$\text{deg}_{A_L}^{-1}(b) = \{x^w : \text{deg}_{A_L}(x^w) = b\} = \mathcal{F}_w = \mathcal{F}_{x^w}.$$
Degree and ordering the fibers

Multiplication by a monomial pushes one fiber inside another fiber.

**Example**

\[ L = \langle (1, -2) \rangle \subset \mathbb{Z}^2, \quad I_L = \langle x - y^2 \rangle. \]

\[ L \text{ is positive. The fibers are finite. We can order the fibers by their } A\text{-degree.} \]
**Proposition**

*When L is positive then all fibers are finite and \( \deg_{A_L} \) determines a partial order on them.*

Caution: we have already seen that this fails if \( L \) is not positive.

**Example**

Let \( L = \mathbb{Z}(1,1) \), \( A = \{1,-1\} \subset \mathbb{Z} \), \( \deg_A(x) = 1 \), \( \deg_A(y) = -1 \), \( \deg_A(xy) = 0 \) and of course the \( A \)-degree of 1 is 0. The fiber that contains 1 is infinite.

\[
\deg_A^{-1}(0) = \{1, xy, \cdots \} = F_1 = F_{x^i y^i}
\]
We consider the $A_L$-degrees of the binomials in a minimal generating set of $I_L$.

**Theorem**

*Let $L \cap \mathbb{N}^m = 0$. All minimal binomial generating sets of $I_L$ have the same cardinality and the same $A_L$-degrees.*

The proof follows from the graded Nakayama Lemma.
Generating $I_L$ when $L \cap \mathbb{N}^m = 0$, CKT (2007).

For every degree $b$ define a subideal of $I_L$ generated by the binomials that have $A$-degrees less than $b$.

**Definition**

$$I_{L,b} = I_{L,F} = (x^u - x^v \mid \deg_{A_L}(x^u) = \deg_{A_L}(x^v) \leq b) \subset I_L$$

where $F$ is the fiber at $b$.

Then we define two graphs.

**Definition**

First graph Let $G(b)$ be the graph with vertices the elements of $\deg_{A_L}^{-1}(b)$ and edges all the sets $\{x^u, x^v\}$ whenever $x^u - x^v \in I_{L,b}$. 
The complete graph on the components of $G(b)$

**Definition**

The second graph is the complete graph with vertex set the connected components of first graph $G(b)$. Let $T_b$ be a spanning tree of this graph.

In this picture, on the left we see the vertices of $G(b)$ while on the right, with red we see a spanning tree $T_b$. 
Spanning trees and generators

For every edge of the tree $T_b$ joining two components of $G(b)$ take one binomial by taking the difference of (two arbitrary) monomials, one from each component.

Let the tree be denoted by red edges: green and orange line segments would produce different sets of binomials for the same tree $T_b$. 
Markov basis for \( I_L \) (CKT07)

For every \( b \), choose a tree \( T_b \) on the graph \( G(b) \) (whose vertices are the connected components of the fiber at \( b \)) and then choose the binomials. Denote this collection by \( \mathcal{F}_{T_b} \).

**Theorem**

The set \( \mathcal{F} = \bigcup_{b \in \mathbb{N}A_L} \mathcal{F}_{T_b} \) is a Markov basis of \( I_L \).

Let \( \mu(I_L) \) be the cardinality of a Markov basis, \( n_b \) be the number of connected components of \( G(b) \), \( t_i(b) \) be the number of vertices of the \( i \)th component.

**Theorem**

\[
\mu(I_L) = \sum_{b \in \mathbb{N}A_L} (n_b - 1)
\]

The number of different Markov bases of \( I_L \) is

\[
\prod_{b \in \mathbb{N}A} t_1(b) \cdots t_{n_b}(b)(t_1(b) + \cdots + t_{n_b}(b))^{n_b-2}
\]
When \( L \cap \mathbb{N}^n \neq 0 \)

It is easier to think of the fibers as the sets:

\[
F_{x^u} = \{ x^u : x^u - x^v \in I \}
\]

**Example**

Let \( L = \langle (1, 1) \rangle \). Then \( I_L = \langle 1 - xy \rangle \). We have seen that \( \deg_{A_L}^{-1}(0) = \{ 1, xy, x^2y^2, \ldots \} = F_1 = F_{xy} \). There are infinitely many fibers.
Example

Let $L = \langle (1, 1), (0, 5) \rangle$. It can be shown that $I = \langle 1 - xy, 1 - x^5 \rangle$.

$$F_1 = \{ x^i y^j : i \equiv j \mod 5 \} = \{ 1, xy, x^5, \ldots \}$$

is represented as the union of black lines. There are five different fibers: $F_1, F_x, F_{x^2}, F_{x^3}, F_{x^4}$. 
Are there fibers ”bigger” than others? would multiplication by monomials determine this order?

Example

Since

$$xF_1 \subset F_x,$$  $$x^4F_x \subset F_{x^5} = F_1$$

the answer seems to be negative.
However we can **define** an equivalence relation on the set of fibers and then we will see that we can **order** the equivalence classes!

**Definition**

\[ F \equiv_L G \iff x^u F \subseteq G, x^v G \subseteq F \]

By $\overline{F}$ we will denote the equivalence class of the fiber $F$.

**Theorem**

*When $L \cap \mathbb{N}^n = \{0\}$ then $\overline{F} = \{F\}$.***
Ordering the fibers

Examples

When $I_L = \langle 1 - xy \rangle$ there is only one equivalence class $F_1 = \{ F_1, F_x, F_y, \ldots \}$ consisting of all (infinitely many) fibers.

When $I = \langle 1 - xy, 1 - x^5 \rangle$ there is only one equivalence class: $F_1 = \{ F_1, F_x, F_x^2, F_x^3, F_x^4 \}$
Ordering equivalence classes of fibers

**Theorem**

All equivalence classes contain the same number of fibers. This number is determined by \( L \cap \mathbb{N}^n \).

Let \( F, G \) be two equivalence classes of \( I_L \)-fibers. We define a partial order on the equivalence classes of the fibers as follows:

\[
F \leq G \text{ if there exists } x^u \text{ such that } x^u F \subset G.
\]

Since \( R \) is Noetherian, an important property is given by the following theorem:

**Theorem**

All chains of equivalence classes of fibers have a minimal element.
Next, we generalize the ideal $I_{L,F}$.

**Definition**

$$I_{L,F} = \langle x^u - x^v \in I : \overline{F_x} < \overline{F} \rangle$$

Let $B = x^u - x^v \in I$ and $G = F_{x^u}$.

**Theorem**

$B$ belongs to a Markov basis of $I_L$ if and only if $B$ is not in $I_{L,F}$. 
Suppose that $L$ is equal to its pure part, that is $\langle L \subset \mathbb{N}^n \rangle$. We have already seen that $F_1$ (the fiber of one) is infinite.

**Theorem**

Assume that the support of $L$ is $n$. All fibers are equivalent to $F_1$.

The following generalizes a result of Sturmfels, Weismantel and Ziegler (95).

**Theorem**

Suppose that the rank of $L$ is $r$. There are $r$ binomials that generate $I_L$ (and thus $I_L$ is a complete intersection).

The theorem actually gives much more. It gives a complete description of all generating sets of $I_L$ setting conditions on the exponents of the binomials.
Two problems arise in the general case while attempting to construct the graphs:

- Infinite fibers.
- In principle we should deal with the fibers in the equivalence classes all at once.

Let $\sigma$ be the support of $L \cap \mathbb{N}^n$. If $F$ is a fiber, we let $m_F$ be the monomial ideal generated by the elements of $F$ where for any variable in $\sigma$ we substitute the value $1$.

**Theorem**

Let $G, F$ be fibers so that $G \equiv F$. Then $m_F = m_G$. 
Let $F$ be a fiber, We construct $G = G(F)$.

**Definition**

First graph Let $G$ be the graph with vertices the set of minimal monomial generators of $m_F$. Two vertices are connected by an edge if the corresponding ”original” binomial is in $I_{L,F}$.

If $F$ and $G$ are two equivalent fibers, then the above construction yields the same graph. We proceed as expected for the second graph.

**Definition**

Second graph: complete graph on the components of $G$. 

Markov Bases of Lattice Ideals
If $L \cap \mathbb{N}^n = \{0\}$ then

- $\sigma = \{\}$
- $F = \{F\}$

the set of minimal monomial generators of $m_F$ is equal to $F$

$I_L, F = I_{L,b}$ where $b$ is the $A$-degree of any element in $F$.

Thus we obtain the same graphs.
Markov Bases of $I_L$

For any fiber $F$ we let $\Gamma(F)$ be the second graph. We consider spanning trees of $\Gamma(F)$.

![Graph 1](image1)

the we make choices of monomials for each red edge (green or orange)

![Graph 2](image2)

then go back and choose a binomial from one of the equivalence classes.
Markov Bases of $I_L$

**Theorem**

A set $S$ of binomials of $I_L$ is a Markov basis of $I_L$ if and only if

- for every $\overline{F}$ the elements of $S$ determine a spanning tree of $\Gamma(\overline{F})$ and
- the binomials of $S$ in the equivalence class of the origin minimally generate the lattice generated by $L \cap \mathbb{N}^n$.

What are the invariants of the Markov bases of $I_L$?

**Theorem**

Let $S = \{B_1, \ldots, B_s\}$ be a Markov basis of $I_L$. The equivalence classes of fibers that correspond to these binomials and their multiplicity in $S$ are uniquely determined and are invariants of $I_L$. 
We can compute the cardinality of a Markov basis, the Markov fibers, the indispensable fibers, the indispensable binomials, and the indispensable monomials. If the rank of $\langle L \cap \mathbb{N}^n \rangle$ is bigger than 1 there are no indispensable binomials. If the rank of $\langle L \cap \mathbb{N}^n \rangle$ is 1, there exists exactly one indispensable binomial.
Cardinality of $I_L$

What is $\mu(I_L)$ the cardinality of a Markov basis?

Let $r$ be the rank of the lattice generated by the elements of $L \cap \mathbb{N}^n$. If $F$ is a fiber we let $t(\overline{F})$ be the number of vertices of $\Gamma_{\overline{F}}$.

Theorem

$$\mu(I_L) = r + \sum_{\overline{F} \neq \{1\}} (t(\overline{F}) - 1),$$


