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Markov Bases of Lattice Ideals

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Lattice Ideals

Let $\Bbbk$ be a field and $R = \Bbbk[x_1, \cdots, x_n]$. For $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ we let $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_n^{u_n}$. Let $L \subset \mathbb{Z}^n$ be a lattice.

Definition

The lattice ideal I_L is

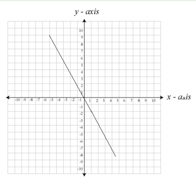
$$I_L := \langle x^u - x^v : \; u - v \in L \rangle = \langle x^{w^+} - x^{w^-} : \; w \in L \rangle$$

The papers mentioned on this lecture are
—, A. Katsabekis, A. Thoma, "Minimal systems of binomial generators and the indispensable complex of a toric ideal", Proc. of the AM.S., 2007.
—, A. Thoma, M. Vladoiou "Markov Bases of Lattice ideals", arXiv: 1303.2303v2

Example: lattice ideals

Example

Let $L \subset \mathbb{Z}^2$ be the lattice generated by $(1, -2)$. $R = \Bbbk[x, y]$



$$(1, -2) = (1, 0) - (0, 2), \quad I_L = \langle x - y^2 \rangle$$

The exponents of the binomials in I_L are in the first quadrant and determine lines parallel to L.

I_L has a unique minimal binomial generating set.

If $L \cap \mathbb{N}^m = \{\mathbf{0}\}$, we say that L is positive.

Example: lattice ideals

Example

Let $L = 3\mathbb{Z}$, so L is generated by 3. We work in $R = \Bbbk[x]$.
$u - v \in L$ iff $u \equiv v \mod 3$.

$$I_L = \langle 1 - x^3 \rangle \ .$$

But also $I_L = \langle 1 - x^6, x^5 - x^8 \rangle$

I_L has (at least) two minimal generating sets of different cardinality.

Minimal generating sets of lattice ideals

Definition

A set S is a Markov basis for I_L if S consists of binomials and S is a minimal generating set of I_L of minimal cardinality.

For counting purposes, a binomial B is the same as $-B$.
How many "different" Markov bases are there? Can we compute the cardinality of a Markov basis? Are there indispensable binomials or indispensable monomial terms of I_L? Is there a characteristic shared by different Markov sets?

Motivation-History

- Paper by Eisenbud and Strumfels: *Binomial ideals*, Duke Math. J. (1996)
- Paper by Diaconis and B. Sturmfels, *Algebraic algorithms for sampling from conditional distributions*, Ann. Statist., (1998)

Applications of Lattice Ideals in Algebraic Statistics, Integer Programming, Graph Theory, Hypergeometric Differential Equations, etc.

Most of the applications deal with the cases of positively graded lattices.

Degrees and fibers

$L \subset \mathbb{Z}^m$. Let $\mathbf{a_i} = \mathbf{e_i} + \mathbf{L}$ for $i = 1, \ldots, m$, $A_L = \{\mathbf{a_1}, \ldots, \mathbf{a_m}\}$,
Recall the A-degree of a monomial

$$\deg_{A_L}(x_1^{u_1} \cdots x_m^{u_m}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \ .$$

and the property

$$I_l = \langle \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} : \ \deg_{A_L}(\mathbf{x}^\mathbf{u}) = \deg_{A_L}(\mathbf{x}^\mathbf{v}) \rangle \ .$$

Yesterday we talked about fibers corresponding to vectors of L ($\mathcal{F}_\mathbf{u}$ or $\mathcal{F}_{\mathbf{u}^+}$). We recall the definition and emphasize the connection to the A-degree.

Definition

Let $b \in \mathbb{N}A_L$. The fiber at b is the following set of monomials:

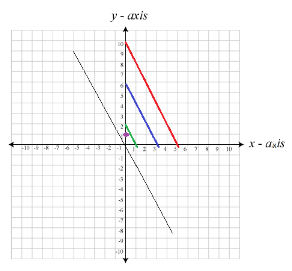
$$\deg_{A_L}^{-1}(\mathbf{b}) = \{\mathbf{x}^\mathbf{w} \ : \ \deg_{A_L}(\mathbf{x}^\mathbf{w}) = \mathbf{b}\} = \mathcal{F}_\mathbf{w} = \mathcal{F}_{\mathbf{x}^\mathbf{w}}$$

Degree and ordering the fibers

Multiplication by a monomial pushes one fiber **inside** another fiber.

Example

$L = \langle (1, -2) \rangle \subset \mathbb{Z}^2$, $I_L = \langle x - y^2 \rangle$.



L is positive. The fibers are finite. We **can** order the fibers by their A-degree.

Fibers and degrees II

Proposition

When L is positive then all fibers are finite and $\deg_{A_L}$ determines a partial order on them.

Caution: we have already seen that this fails if L is not positive.

Example

Let $L = \mathbb{Z}(1,1)$, $A = \{1, -1\} \subset \mathbb{Z}$, $\deg_A(x) = 1$, $\deg_A(y) = -1$ $\deg_A(xy) = 0$ and of course the A-degree of 1 is 0. The fiber that contains 1 is infinite.

$$\deg_A^{-1}(0) = \{1, xy, \cdots\} = F_1 = F_{x^i y^i}$$

Invariants when $L \cap \mathbb{N}^m = 0$.

We consider the A_L-degrees of the binomials in a minimal generating set of I_L.

Theorem

Let $L \cap \mathbb{N}^m = 0$. All minimal binomial generating sets of I_L have the same cardinality and the same A_L-degrees.

The proof follows from the graded Nakayama Lemma.

Generating I_L when $L \cap \mathbb{N}^m = 0$, CKT (2007).

For every degree b define a subideal of I_L generated by the binomials that have A-degrees **less** than b.

Definition

$$I_{L,\mathbf{b}} = I_{L,F} = (\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \deg_{A_L}(\mathbf{x}^{\mathbf{u}}) = \deg_{A_L}(\mathbf{x}^{\mathbf{v}}) \lneqq \mathbf{b}) \subset I_L$$

where F is the fiber at $\mathbf{b}$.

Then we define two graphs.

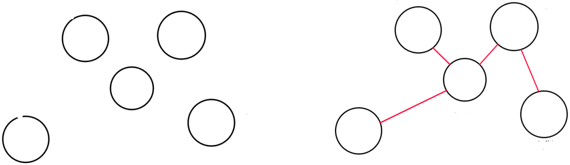
Definition

First graph Let $G(\mathbf{b})$ be the graph with vertices the elements of $\deg_{A_L}^{-1}(\mathbf{b})$ and edges all the sets $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ whenever $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{L,\mathbf{b}}$.

The complete graph on the components of $G(\mathbf{b})$

Definition

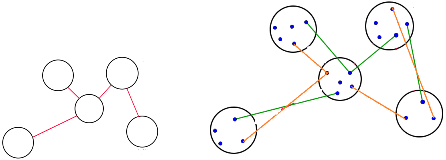
The <u>second graph</u> is the complete graph with vertex set the connected components of first graph $G(\mathbf{b})$. Let $T_{\mathbf{b}}$ be a spanning tree of this graph.



In this picture, on the left we see the vertices of $G(\mathbf{b})$ while on the right, with red we see a spanning tree $T_{\mathbf{b}}$

Spanning trees and generators

For every edge of the tree $T_{\mathbf{b}}$ joining two components of $G(\mathbf{b})$ take one binomial by taking the difference of (two arbitrary) monomials, one from each component.



Let the tree be denoted by red edges: green and orange line segments would produce different sets of binomials for the same tree $T_{\mathbf{b}}$.

Markov basis for I_L (CKT07)

For every $\mathbf{b}$, choose a tree $T_{\mathbf{b}}$ on the graph $G(\mathbf{b})$ (whose vertices are the connected components of the fiber at $\mathbf{b}$) and then choose the binomials. Denote this collection by $\mathcal{F}_{T_{\mathbf{b}}}$.

Theorem

The set $\mathcal{F} = \cup_{\mathbf{b}\in\mathbb{N}A_L}\mathcal{F}_{T_{\mathbf{b}}}$ *is a Markov basis of* I_L.

Let $\mu(I_L)$ be the cardinality of a Markov basis, $n_{\mathbf{b}}$ be the number of connected components of $G(\mathbf{b})$, $t_i(\mathbf{b})$ be the number of vertices of the ith component.

Theorem

$$\mu(I_L) = \sum_{\mathbf{b}\in\mathbb{N}A_L} (n_{\mathbf{b}} - 1)$$

The number of different Markov bases of I_L is

$$\prod_{\mathbf{b}\in\mathbb{N}A} t_1(\mathbf{b})\cdots t_{n_{\mathbf{b}}}(\mathbf{b})(t_1(\mathbf{b}) + \cdots + t_{n_{\mathbf{b}}}(\mathbf{b}))^{n_{\mathbf{b}}-2}$$

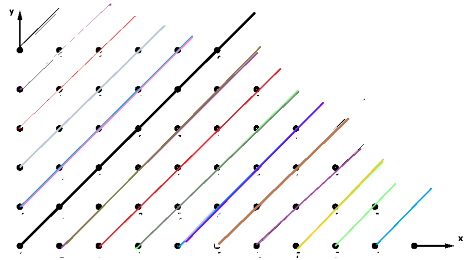
When $L \cap \mathbb{N}^n \neq 0$

It is easier to think of the fibers as the sets:

$$F_{\mathbf{x^u}} = \{\mathbf{x^u} : \ \mathbf{x^u} - \mathbf{x^v} \in I\}$$

Example

Let $L = \langle(1,1)\rangle$. Then $I_L = \langle 1 - xy\rangle$. We have seen that $\deg_{A_L}^{-1}(0) = \{1, xy, x^2y^2, \ldots\} = F_1 = F_{xy}$. There are infinitely many fibers.



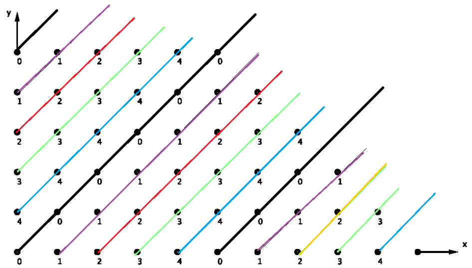
Infinite Fibers

Example

Let $L = \langle (1,1), (0,5) \rangle$. It can be shown that $I = \langle 1 - xy, 1 - x^5 \rangle$.

$$F_1 = \{x^i y^j : \ i \equiv j \mod 5\} = \{1, xy, x^5, \ldots\}$$

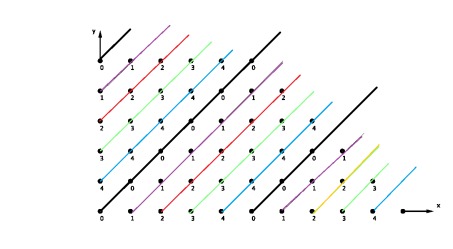
is represented as the union of black lines. There are five different fibers: $F_1, F_x, F_{x^2}, F_{x^3}, F_{x^4}$.



Ordering the fibers

Are there fibers "bigger" than others? would multiplication by monomials determine this order?

Example



Since

$$xF_1 \subset F_x, \quad x^4 F_x \subset F_{x^5} = F_1$$

the answer seems to be negative.

Ordering the fibers II

However we can **define** an equivalence relation on the set of fibers and then we will see that we can order the equivalence classes!

Definition

$$F \equiv_L G \Leftrightarrow x^u F \subset G, x^v G \subset F$$

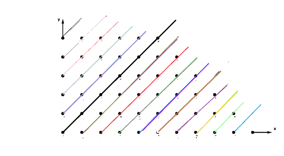
By $\overline{F}$ we will denote the equivalence class of the fiber F.

Theorem

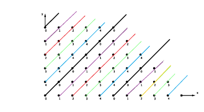
When $L \cap \mathbb{N}^n = \{0\}$ *then* $\overline{F} = \{F\}$.

Ordering the fibers

Examples



When $I_L = \langle 1 - xy \rangle$ there is only one equivalence class $\overline{F_1} = \{F_1, F_x, F_y, \ldots\}$ consisting of all (infinitely many) fibers.



When $I = \langle 1 - xy, 1 - x^5 \rangle$ there is only one equivalence class:

$$\overline{F_1} = \{F_1, F_x, F_{x^2}, F_{x^3}, F_{x^4}\}$$

Ordering equivalence classes of fibers

Theorem

All equivalence classes contain the same number of fibers. This number is determined by $L \cap \mathbb{N}^n$.

Let $\overline{F}$, $\overline{G}$ be two equivalence classes of I_L-fibers. We define a partial order on the equivalence classes of the fibers as follows:

$$\overline{F} \leq \overline{G} \text{ if there exists } x^u \text{ such that } x^u F \subset G \ .$$

Since R is Noetherian, an important property is given by the following theorem:

Theorem

All chains of equivalence classes of fibers have a minimal element.

Generating a lattice ideal I_L

Next, we generalize the ideal $I_{L,F}$.

Definition

$$I_{L,\ \overline{F}} = \langle x^u - x^v \in I : \ \overline{F_{x^u}} < \overline{F} \rangle$$

Let $B = x^u - x^v \in I$ and $G = F_{x^u}$.

Theorem

B belongs to a Markov basis of I_L if and only if B is not in $I_{L,\ \overline{F}}$.

Pure lattices

Suppose that L is equal to its pure part, that is $\langle L \subset \mathbb{N}^n \rangle$. We have already seen that F_1 (the fiber of one) is infinite.

Theorem

Assume that the support of L is n. All fibers are equivalent to F_1.

The following generalizes a result of Sturmfels, Weismantel and Ziegler (95).

Theorem

Suppose that the rank of L is r. There are r binomials that generate I_L (and thus I_L is a complete intersection)

The theorem actually gives much more. It gives a complete description of all generating sets of I_L setting conditions on the exponents of the binomials.

Generalizing CKT

Two problems arise in the general case while attempting to construct the graphs:

- Infinite fibers.
- In principle we should deal with the fibers in the equivalence classes all at once.

Let σ be the support of $L \cap \mathbb{N}^n$. If F is a fiber, we let m_F be the monomial ideal generated by the elements of F where for any variable in σ we substitute the value 1.

Theorem

Let G, F be fibers so that $G \equiv F$. Then $m_F = m_G$.

Generalizing the graphs on the fibers

Let F be a fiber, We construct $G = G(F)$.

Definition

First graph Let G be the graph with vertices the set of minimal monomial generators of m_F. Two vertices are connected by an edge if the corresponding "original" binomial is in $I_{L,\overline{F}}$.

If F and G are two equivalent fibers, then the above construction yields the same graph. We proceed as expected for the second graph.

Definition

Second graph: complete graph on the components of G.

True Generalization

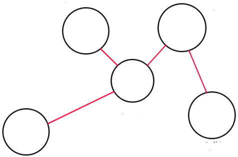
If $L \cap \mathbb{N}^n = \{0\}$ then

- $\sigma = \{\}$
- $\overline{F} = \{F\}$
- the set of minimal monomial generators of m_F is equal to F
- $I_{L, \overline{F}} = I_{L,b}$ where b is the A-degree of any element in F.

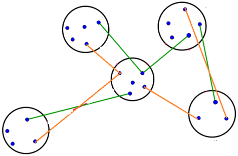
Thus we obtain the same graphs.

Markov Bases of I_L

For any fiber F we let $\Gamma(\overline{F})$ be the second graph. We consider spanning trees of $\Gamma(\overline{F})$.



the we make choices of monomials for the each red edge (green or orange)



then go back and choose a binomial from one of the equivalence classes

Markov Bases of I_L

Theorem

A set S of binomials of I_L is a Markov basis of I_L if and only if

- *for every $\overline{F}$ the elements of S determine a spanning tree of $\Gamma(\overline{F})$ and*
- *the binomials of S in the equivalence class of the origin minimally generate the lattice generated by $L \cap \mathbb{N}^n$.*

What are the invariants of the Markov bases of I_L?

Theorem

Let $S = \{B_1, \dots, B_s\}$ be a Markov basis of I_L. The equivalence classes of fibers that correspond to these binomials and their multiplicity in S are uniquely determined and are invariants of I_L.

Markov Bases of I_L

We can compute the cardinality of a Markov basis, the Markov fibers, the indispensable fibers, the indispensable binomials, and the indispensable monomials. If the rank of $\langle L \cap \mathbb{N}^n \rangle$ is bigger than 1 there are no indispensable binomials. If the rank of $\langle L \cap \mathbb{N}^n \rangle$ is 1, there exists exactly one indispensable binomial.

Cardinality of I_L

What is $\mu(I_L)$ the cardinality of a Markov basis?

Let r be the rank of the lattice generated by the elements of $L \cap \mathbb{N}^n$. If F is a fiber we let $t(\overline{F})$ be the number of vertices of $\Gamma_{\overline{F}}$.

Theorem

$$\mu(I_L) = r + \sum_{\overline{F} \neq \overline{F}_{\{1\}}} (t(\overline{F}) - 1),$$

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