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Free Resolutions of Lattice Ideals

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Lattice ideals: the general setting

We let $L \subset \mathbb{Z}^n$ be a lattice and $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ where $\mathbf{a}_i = e_i + L$. We also let $\Bbbk$ be a field, $R = \Bbbk[x_1, \ldots, x_n]$ and A-grade the monomials: for $u \in \mathbb{N}^n$ we let

$$\deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m$$

The lattice ideal is

$$I_L := \langle\, \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \; \deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}}) \,\rangle$$

$$= \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \; u \in L \rangle$$

We let $L_{pure} = \langle L \cap \mathbb{N}^n \rangle$.

Example of complete intersection

Example

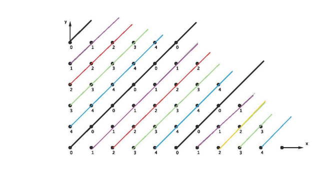
Let $v_1 = (3,0,1,-1,0)$, $v_2 = (0,1,6,0,-1)$, $v_3 = (1,1,0,0,0)$, $v_4 = (5,0,0,0,0)$, $v_5 = (0,5,0,0,0)$ and $L = \langle v_1, \dots, v_4 \rangle$. Then

$$L \cap \mathbb{N}^5 = \mathbb{N}v_3 + \mathbb{N}v_4 + \mathbb{N}v_5, \quad L_{pure} = \langle L \cap \mathbb{N}^5 \rangle = \mathbb{Z}v_3 + \mathbb{Z}v_4,$$

$$\operatorname{rank}(L_{pure}) = 2, \quad \sigma = \{1,2\},$$

$$I_{L_{pure}} = \langle 1 - x_1^5, 1 - x_1 x_2 \rangle.$$

We studied $I_{L_{pure}}$ yesterday. We recall the picture of its fibers.



Example of c.i. (continued)

Example

The following can be shown:

- $\overline{F_1}$ consists of 5 equivalent fibers
- I_L is generated by 4 binomials.
- A Markov basis of I_L must contains 2 binomials associated to $\overline{F_1}$, 1 binomial associated to $\overline{F_{x_4}}$ and 1 binomial associated to $\overline{F_{x_5}}$.

Since rank $L = 4$ and I_L is generated by 4 binomials, I_L is a complete intersection.

We note that if L is a lattice ideal of rank r then I_L needs at least r binomial generators.

Binomial Complete Intersection

Definition

Let L be a lattice of rank r. The lattice ideal I_L is called a *binomial complete intersection* if there exist binomials $B_1, \ldots, B_r$ such that $I_L = \langle B_1, \ldots, B_r \rangle$.

When the graded Nakayama's lemma applies (i.e. when $L \cap \mathbb{N}^n = \{\mathbf{0}\}$) then complete intersection lattice ideals are automatically binomial complete intersections.

Are there criteria for a lattice ideal to be a complete intersection ideal?

History of c.i when $L \cap \mathbb{N}^n = \{0\}$.

The problem is completely solved when L is positively graded by a series of articles: Herzog (70), Delorne(76), Stanley (77), Ishida (78), Watanabe(80), Nakajima(85), Schafer (85), Rosales and Garcia-Sanchez (95), Fischer, Morris and Shapiro (95), Scheja, Scheja and Storch (99), Morales and Thoma (05).

The final conclusion of this series of articles is that when L is a positive lattice, the lattic ideal I_L is a complete intersection if and only if the matrix M whose rows correspond to a basis of L is **mixed dominating**.

Mixed dominating matrices

Definition

A matrix M is mixed dominating if every row of M has a positive and negative entry and M contains no square submatrix with this property.

Example

$$\begin{bmatrix} 1 & -1 & 0 \\ 6 & 0 & -1 \end{bmatrix}$$

Rank of the lattice L

Recall that $L_{pure} = \langle L \cap \mathbb{N}^n \rangle$. Let $\sigma = \text{supp}(L_{pure})$.

- If $u \in \mathbb{Z}^n$, by u^σ we mean the vector $(u_i)_{i \notin \sigma}$.
- By L^σ we mean the lattice generated by all u^σ, $u \in L$.

Applying linear algebra one can show as in
–, Thoma, Vladoiu, *Markov bases of lattice ideals*, preprint (CTV14) the following:

Theorem

$$\text{rank}(L) = \text{rank}(L^\sigma) + \text{rank}(L_{pure}).$$

Example for the rank equation

Example

In this example we write the generators of L as rows of a matrix and then we compute L^σ.

$$L: \quad \begin{bmatrix} 3 & 0 & 1 & -1 & 0 \\ 0 & 1 & 6 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\sigma = \{1, 2\}$ it follows that

$$L^\sigma: \quad \begin{bmatrix} 1 & -1 & 0 \\ 6 & 0 & -1 \end{bmatrix}$$

Note that L^σ is mixed dominating matrix.

Complete intersections and mixed dominating matrices in the general case

From previous slide recall that

$$\mathrm{rank}(L) = \mathrm{rank}(L^\sigma) + \mathrm{rank}(L_{pure}).$$

In (CTV14) we also prove the following:

Theorem

$$\mu(I_L) = \mu(I_{L^\sigma}) + \mathrm{rank}(L_{pure}) \ .$$

Thus

Theorem

(CTV14) Let $L \subset \mathbb{Z}^n$ be a lattice. The ideal I_L is binomial complete intersection if and only if there exists a basis of L^σ so that its vectors give the rows of a mixed dominating matrix.

Minimal free A-graded Resolutions of R/I_L

We consider A-graded free modules F_i, i.e.$F_i \cong \oplus R[-\mathbf{b}]_i$ where for $b \in \mathbb{N}A$, $R[-\mathbf{b}] := R \cdot e$. The exact sequence

$$(\mathbf{F}_{\mathcal{L}}, \phi) : \quad 0 \longrightarrow F_p \xrightarrow{\phi_p} \cdots\cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow R/I_L \longrightarrow 0,$$

is an A-graded **minimal** free resolution of $R/I_{\mathcal{L}}$ if

$$\phi_i(F_i) \subset (x_1, \ldots, x_n)F_{i-1} \ .$$

We let $\beta_{i,\mathbf{b}}(R/I_L)$ be the rank of the F_i. It is well known that

$$\beta_{i,\mathbf{b}}(R/I_L) = \dim_{\mathbb{k}} \mathrm{Tor}_i(R/I_L, \mathbb{k})_{\mathbf{b}}$$

Simple syzygies

The elements of $\ker \phi_i$ are called ***i*-syzygies**. To describe a resolution we need to fix basis elements for all F_i:
Let $\{E_t : \; t = 1, \ldots, s_i\}$ be a basis of F_i and

$$h = \sum_{1 \leq t \leq s_i} (\sum_{c_{\mathbf{a_t}} \neq 0} c_{\mathbf{a_t}} \mathbf{x}^{\mathbf{a_t}}) E_t$$

be an i-syzygy. We want resolutions that are **simple.** For example the syzygies on the 0th stage **should** be the binomial generators of the ideal.

- what is the correct correspondence to the notion of binomial generator in terms of syzygies?
- what does indispensable i-syzygy mean?

Simple syzygies

If

$$h = \sum_{1 \leq t \leq s_i} \Big(\sum_{c_{\mathbf{a}_t} \neq 0} c_{\mathbf{a}_t} \mathbf{x}^{\mathbf{a}_t} \Big) E_t$$

we let

$$S(h) = \{ \mathbf{x}^{\mathbf{a}_t} E_t : \; c_{\mathbf{a}_t} \neq 0 \}.$$

In –, A. Thoma, *On simple A-multigraded minimal resolutions*, Contemp. Math. (CT09) we gave the following definition:

Definition

h is **simple** syzygy if there is no nonzero $h' \in \ker \phi_i$ such that $S(h') \subsetneqq S(h)$.

Simple resolutions

Let $(\mathbf{F}_\bullet, \phi)$ be a minimal free resolution of R/I_L

Theorem

There exists a system of bases $\mathbf{B}$ *for the* F_i *so that* $(\mathbf{F}_\bullet, \phi, \mathbf{B})$ *is simple.*

Assume now that $L \cap \mathbb{N}^n = \{0\}$. Fix the system of bases $\mathbf{B}$ and identify for all practical purpose the syzygies that differ by a constant multiple. In (CT09) we showed the following:

Theorem

The set of minimal simple i*-syzygies of* R/I *that generate* $\ker \phi_i$ *is finite.*

Indispensable complexes

Definition

We say that $(\mathbf{F}_\bullet, \phi, \mathbf{B})$ is an *indispensable complex* of R/I_L if for each minimal simple free resolution $(\mathbf{G}_\bullet, \theta, \mathbf{C})$ of R/I_L where $C_0 = \{1\}$, there is an injective homomorphism $\omega : (\mathbf{F}_\bullet, \phi, \mathbf{B}) \to (\mathbf{G}_\bullet, \theta, \mathbf{C})$ such that $\omega_0 = id_R$.

In particular the elements of F_1 determine indispensable minimal binomial generators of I.

When is the Koszul comlex indispensable?

Assume now that $L_{pure} = \{0\}$.

Theorem

Let $I_{\mathcal{L}} = \langle f_1, \ldots, f_s \rangle$ be a lattice ideal where $\{f_i : i = 1, \ldots, s\}$ is an R-sequence of binomials such that $\mathbf{b}_i = \deg_A(f_i)$ are **incomparable***. The Koszul complex on the f_i is an indispensable complex of $R/I_{\mathcal{L}}$.*

Such examples of ideals are given by complete intersection ideals of bipartite graphs.

Projective dimension

Let $L \subset \mathbb{Z}^n$ be such that $L \cap \mathbb{N}^n = \{0\}$. In 1998, I. Peeva and B. Sturmfels, Math.Z. showed that

Theorem

The projective dimension of R/I_L is at most $2^{\mathrm{rank}(L)} - 1$.

The proof is based on estimating the homology of $\Delta_{\mathbf{b}}$ for $b \in \mathbb{N}A$. Recall that the fiber at b is the **finite** set of monomials

$$F_{\mathbf{b}} := \{x^u : \deg_A(x^u) = \mathbf{b}\}$$

The simplicial complex

$$\Delta_{\mathbf{b}} := \{\tau \subset \{1, \ldots, n\} : \exists\, x^{\mathbf{a}} \in F_{\mathbf{b}},\ \tau \subset \operatorname{supp} x^{\mathbf{a}}\}$$

Theorem

$$\beta_{i,\mathbf{b}}(R/I_L) = \dim_{\Bbbk} \overline{\mathrm{H}_i}(\Delta_{\mathbf{b}})$$

The gcd-complex

Let $b \in \mathbb{N}A$, $F_{\mathbf{b}} := \{x^u : \deg_A(x^u) = \mathbf{b}\}$. We consider instead the simplicial complex $\Delta_{\text{gcd}}(\mathbf{b})$ with vertices the elements of $F_{\mathbf{b}}$ and faces all subsets $T \subset F_{\mathbf{b}}$ such that $\gcd(x^{\mathbf{a}} : x^{\mathbf{a}} \in T) \neq 1$.

It was shown independently by (CT09) and Ojeda, Vigneron-Tenorio (2010) that

$\Delta_{\text{gcd}}(\mathbf{b})$ and $\Delta_{\mathbf{b}}$ have the same homology.

Theorem

$$\beta_{i,\mathbf{b}}(R/I_L) = \dim_{\Bbbk} \overline{\mathrm{H}_i}(\Delta_{\text{gcd}}(\mathbf{b}))$$

Example

$$A = \begin{bmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

$$I_L = (bc - ad, ac^2 - b^2 d, b^3 - a^2 c, c^3 - bd^2)$$

$$(\mathbf{F}_{\mathcal{L}}, \phi) : \quad \mathbf{0} \to \mathbf{R} \to \mathbf{R^4} \to \mathbf{R^4} \to \mathbf{R} \longrightarrow \mathbf{R}/\mathbf{I_L} \to \mathbf{0}$$

The generators of I_L have degrees $(4,4)$, $(6,6)$, $(9,3)$, $(3,9)$. The 2-betti degrees are:

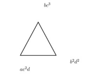
$$(6,10),\ (7,9),\ (9,7),\ (10,6)$$

while the 3-betti degree is $(10,10)$.
In the next slide we see the fibers and the corresponding simplicial complexes at the degrees $(4,4)$, $(6,10)$, $(10,10)$.
Is this resolution indispensable?

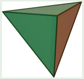
Example

$$C_{(4,4)} = \{ad, bc\}$$

$C_{(6,10)} = \{bc^3, ac^2d, b^2d^2\}$,



$C_{(10,10)} =$
$\{abc^3, a^2c^2d, ab^2d^2, b^3cd\}$



Indispensable binomials

Question How can we find the indispensable binomials?
Answer If **b** is minimal and the gcd simplicial complex at degree **b** consists of two points, then the difference of the two monomial/points is an indispensable binomial.

Free resolutions

- D. Bayer, B. Sturmfels, *Cellular resolutions of monomial modules*, J. Reine Angew. Math. **502** (1998), 123-140.
- I. Peeva and B. Sturmfels, *Generic Lattice Ideals*, J. Amer. Math. Soc. **11** (1998) 363-373.
- I. Peeva and B. Sturmfels, *Syzygies of codimension 2 lattice ideals*, Math Z. **229** (1998) no 1, 163-194.

Algebraic Scarf complexes

In 1998 Peeva and Sturmfels considered the fibers $C_{\mathbf{b}}$ such that $\gcd(C_{\mathbf{b}}) = 1$ and $\gcd(C_{\mathbf{b}} \setminus \{m\}) \neq 1$ for all $m \in C_{\mathbf{b}}$. They called them **basic.** The algebraic **Scarf complex** was then defined as follows:

$$(\mathbf{G}, \theta) \ := \bigoplus_{G} R \cdot E_G$$

G denotes a basic fibers, so th sum runs over all basic fibers. E_G denotes a basis vector in homological degree $|G| - 1$.

$$\theta(E_G) = \sum_{m \in G} \text{sign}(m, G) \gcd(G \setminus \{m\}) E_{[G \setminus \{m\}]},$$

They showed that this complex is indispensable and when the binomial generators of I_L have full support (i.e. I_L is generic) then the Scarf complex is a minimal resolution.

Taylor resolution

The Scarf complex is **not** always a resolution of S/I_L.

In 1998 Bayer and Sturmfels gave the **Taylor resolution** of S/I_L. This resolution is of similar form

$$(\mathbf{G}, \theta) \ := \bigoplus_{G} R \cdot E_G$$

with the same differential

$$\theta(E_G) = \sum_{m \in G} \mathrm{sign}(m, G) \, \mathrm{gcd}(G \setminus \{m\})$$

except that the set G which shows up in the ith homological position is not a basic fiber, but a subset of anyfiber so that $|G| = i + 1$ and has gcd equal to 1. This is not a always a minimal resolution.

Components of fibers

In –, Thoma *On the generalized Scarf complex of lattice ideals* (CT10) we considered **basic components of fibers.**

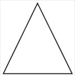
Definition

Let $C_{\mathbf{b}}$ be a fiber and $G \subset C_{\mathbf{b}}$. We say that G is a **basic component** of $C_{\mathbf{b}}$ if the following are satisfied:

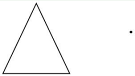
- $\gcd(G) = 1$.
- $\gcd(G \setminus \{m\}) \neq 1$ for all $m \in G$ and
- G is a connected component of $\Delta_{\gcd}(\mathbf{b})$ if $|C_{\mathbf{b}}| > 2$.

Examples

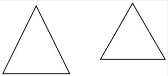
- The fiber itself is basic
$$C_b = \{bc^3, ac^2d, b^2d^2\}$$



- The basic components are a triangle and a point
$$C_b = \{abd, ac^2, b^2c, e^2\}$$



- The basic components are two triangles
$$C_B = \{a^3c, bc^2, a^2b^2, e^2f, f^2, d^3e\}$$



Generalized Algebraic Scarf complex

The generalized algebraic Scarf complex, i.e.

$$(\mathbf{G_L}, \theta) \; := \bigoplus_{G} R \cdot E_G$$

is defined as the algebraic Scarf complex, except that the sum runs over all G which are basic components of fibers (instead of basic fibers) and has the same differential as the Taylor complex and the alg. Scarf complex.

Theorem

The generalized algebraic Scarf complex is is indispensable for $R/I_{\mathcal{L}}$.

The complex might be exact even when the ideal is not generic.

Theorem

If $I_{\mathcal{L}}$ is generated by indispensable binomials then the alg. Scarf complex equals the gen. alg. Scarf complex.

Cases of Indispensable complexes

In (CT09) the notion of **strongly indispensable** was also given.

Barrucci, Froberg, Sahin recently in (BFS14) discuss minimal free resolutions of monomial curves when R/I_L is Gorenstein and the codimension is small. Moreover they examine when these resolutionss are **strongly** indispensable.