On the resolution of binomial edge ideals

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Outline

- Objects
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- Problems and results
Simple graphs: undirected, no loops, no multiple edges
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\[ G : V(G) = [n], E(G) \]
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Binomial ideals $\subset S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$
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The binomial edge ideal of $G$
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The binomial edge ideal of $G$


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First examples

- $G = K_n$, $J_G = (f_{ij} : 1 \leq i < j \leq n) = l_2(X)$ where

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$
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- $G = K_n$, $J_G = (f_{ij} : 1 \leq i < j \leq n) = I_2(X)$ where
  \[
  X = \begin{pmatrix}
  x_1 & x_2 & \cdots & x_n \\
  y_1 & y_2 & \cdots & y_n
  \end{pmatrix}.
  \]

- $G = L_n$, $J_G = (f_{i,i+1} : 1 \leq i \leq n-1)$, the ideal of adjacent minors of $X$. 

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On the resolution of binomial edge ideals
Theorem (HHHKR)

Let $G$ be a simple graph on the vertex set $[n]$ with the edge set $E(G)$, and let $<$ be the lexicographic order on $S$ induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Then the following conditions are equivalent:

(a) The generators $f_{ij}$ of $J_G$ form a quadratic Gröbner basis.

(b) For all edges $\{i, j\}$ and $\{i, k\}$ with $j > i < k$ or $j < i > k$ one has $\{j, k\} \in E(G)$. 
Closed graphs

1. Closed
   - 1 → 2 → 3
   - 2 → 3

2. Not closed
   - 2 → 1 → 3
Closed graphs

\[ \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array} \]
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On the resolution of binomial edge ideals
Closed graph = there exists a labeling of its vertices with respect to which it is closed.
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\[ G \text{ closed} \implies G \text{ chordal and claw-free.} \]
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On the resolution of binomial edge ideals
Closed graphs with CM binomial edge ideal: Betti numbers

Theorem (Ene, Herzog, Hibi)

Let $G$ be a closed graph with Cohen–Macaulay binomial edge ideal. Then $\beta_{ij}(J_G) = \beta_{ij}(\text{in}(J_G))$ for all $i$ and $j$. 
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Conjectures:

- $\beta_{ij}(J_G) = \beta_{ij}(\text{in}_<(J_G))$ for all $i$ and $j$ for any closed graph $G$. 
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Let $G$ be a closed graph with Cohen–Macaulay binomial edge ideal. Then $\beta_{ij}(J_G) = \beta_{ij}(\text{in}(J_G))$ for all $i$ and $j$.

Conjectures:

- $\beta_{ij}(J_G) = \beta_{ij}(\text{in}_<(J_G))$ for all $i$ and $j$ for any closed graph $G$.
- The extremal Betti numbers of $J_G$ and $\text{in}_<(J_G)$ coincide for any graph $G$. 
Example
### Example

<table>
<thead>
<tr>
<th></th>
<th>$J_G$</th>
<th>$\text{in}(J_G)$</th>
</tr>
</thead>
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<tr>
<td></td>
<td>0 1 2 3 4</td>
<td>0 1 2 3 4</td>
</tr>
<tr>
<td>0:</td>
<td>1 – – – – –</td>
<td>1 – – – – – –</td>
</tr>
<tr>
<td>1:</td>
<td>– 6 4 – –</td>
<td>– 6 5 – – –</td>
</tr>
<tr>
<td>2:</td>
<td>– – 9 12 4</td>
<td>– 1 10 12 4</td>
</tr>
<tr>
<td>Total:</td>
<td>1 6 13 12 4</td>
<td>1 7 15 13 4</td>
</tr>
</tbody>
</table>

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A. Dokuyucu, *Extremal Betti numbers of some classes of binomial edge ideals*, accepted, The Mathematical Reports.
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Betti diagram of $S/J_G$ where $G = K_{m,n}$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$mn$</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$\beta_{24}$</td>
<td>...</td>
<td>$\beta_{p,p+2}$</td>
</tr>
</tbody>
</table>

where $p = \text{proj dim } S/J_G = \begin{cases} m, & \text{if } n = 1, \\ 2m + n - 2, & \text{if } n > 1. \end{cases}$

\[ S/J_G \text{ has a unique extremal Betti number, namely } \]

\[ \beta_{p,p+2} = \begin{cases} 
  m-1, & \text{if } p = m, \\
  n-1, & \text{if } p = 2m+n-2.
\end{cases} \]
Theorem (HHHKR)

The set of binomials

\[ \mathcal{G} = \bigcup_{i<j} \{ u_\pi f_{ij} : \pi \text{ is an admissible path from } i \text{ to } j \} \]

is the reduced Gröbner basis of \( J_G \) with respect to lexicographic order on \( S \) induced by the natural order of indeterminates, \( x_1 > \cdots > x_n > y_1 > \cdots > y_n \).
Gröbner basis [HHHKR]

**Definition**

Let $i < j$ be two vertices of $G$. A path $i = i_0, i_1, \ldots, i_{r-1}, i_r = j$ from $i$ to $j$ is called **admissible** if the following conditions are satisfied:

(i) $i_k \neq i_\ell$ for $k \neq \ell$;

(ii) for each $k = 1, \ldots, r - 1$, one has either $i_k < i$ or $i_k > j$;

(iii) for any proper subset $\{j_1, \ldots, j_s\}$ of $\{i_1, \ldots, i_{r-1}\}$, the sequence $i, j_1, \ldots, j_s, j$ is not a path in $G$.

With a given admissible path $\pi$ of $G$ from $i$ to $j$, we associate a monomial

$$u_\pi = (\prod_{i_k > j} x_{i_k})(\prod_{i_\ell < i} y_{i_\ell}),$$

where $\pi : i = i_0, i_1, \ldots, i_{r-1}, i_r = j$, from $i$ to $j$, $i < j$. 
Let $G = K_{3,2}$ be a complete bipartite graph with 5 vertices.

The admissible paths of $K_{3,2}$ other than the edges are:

$$
\pi_1 = 1, 4, 2; \quad \pi_2 = 1, 5, 2; \quad \pi_3 = 1, 4, 3; \quad \pi_4 = 1, 5, 3; \quad \pi_5 = 2, 4, 3; \\
\pi_6 = 2, 5, 3; \quad \pi_7 = 4, 1, 5; \quad \pi_8 = 4, 2, 5; \quad \pi_9 = 4, 3, 5.
$$
Let $G = K_{3,2}$ be a complete bipartite graph with 5 vertices.

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\[ f_{14}, f_{15}, f_{24}, f_{25}, f_{34}, f_{35}, \]

\[ x_4 f_{12}, x_5 f_{12}, x_4 f_{13}, x_5 f_{13}, x_4 f_{23}, x_5 f_{23}, y_1 f_{45}, y_2 f_{45}, y_3 f_{45} \]
Corollary

Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $V(G) = \{1, \ldots, m\} \cup \{m + 1, \ldots, m + n\}$. Then

$$\text{in}_<(J_G) = (\{x_i y_j\}_{1 \leq i \leq m}^{m+1 \leq j \leq m+n}, \{x_i x_{m+k} y_j\}_{1 \leq i < j \leq m}, \{x_{m+i} y_k y_{m+j}\}_{1 \leq i < j \leq n}^{1 \leq k \leq m}).$$
Theorem

Let $G = K_{m,n}$ be the complete graph. Then

(a) $\text{in}_<(J_G)$ has linear quotients.

(b) The graded Betti numbers of $\text{in}_<(J_G)$ are:

$$\beta_{t,t+2}(\text{in}_<(J_G)) = \sum_{1 \leq i \leq m, m+1 \leq j \leq m+n} \binom{i+j-m-2}{t},$$

$$\beta_{t,t+3}(\text{in}_<(J_G)) = \begin{cases} 
\sum_{1 \leq i < j \leq m, 1 \leq k \leq n} \binom{n+k+j-3}{t}, & \text{if } n = 1, \\
\sum_{1 \leq i < j \leq m, 1 \leq k \leq n} \binom{n+k+j-3}{t} + \sum_{1 \leq i < j \leq n, 1 \leq k \leq m} \binom{m+k+j-3}{t}, & \text{if } n > 1. 
\end{cases}$$
Corollary

Let $G = K_{m,n}$ be the complete graph. Then:

(a) $\text{projdim}(S/\text{in}_<(J_G)) = \text{projdim}(\text{in}_<(J_G)) + 1 = \begin{cases} m, & \text{if } n = 1, \\ 2m + n - 2, & \text{if } n > 1. \end{cases}$

(b) $S/\text{in}_<(J_G)$ has a unique extremal Betti number, namely

$$\beta_{p,p+2}(S/\text{in}_<(J_G)) = \beta_{p-1,p+2}(\text{in}_<(J_G)) = \begin{cases} m - 1, & \text{if } n = 1, \\ n - 1, & \text{if } n > 1. \end{cases}$$
$C_n, n \geq 4$


\[
\begin{array}{ccccccc}
 & 0 & 1 & 2 & 3 & \cdots & n \\
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & n & 0 & 0 & \cdots & 0 \\
2 & 0 & 0 & \beta_{24} & 0 & \cdots & 0 \\
3 & 0 & 0 & 0 & \beta_{36} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n-2 & 0 & 0 & \beta_{2,n} & \beta_{3,n+1} & \cdots & \beta_{n,2n-2}
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & \cdots & n \\
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & n & 0 & 0 & \cdots & 0 \\
2 & 0 & 0 & \beta_{24} & 0 & \cdots & 0 \\
3 & 0 & 0 & 0 & \beta_{36} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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\end{array}
\]

\[
\beta_{n,2n-2} = \binom{n-1}{2} - 1.
\]
Admissible paths in $C_n$: edges $i, i+1$ for $1 \leq i \leq n-1$ and $1, n$ together with $i, i-1, \ldots, 1, n, n-1, \ldots, j+1, j$ where $2 \leq j - i \leq n-2$. 
Admissible paths in $C_n$: edges $i, i+1$ for $1 \leq i \leq n-1$ and $1, n$ together with $i, i-1, \ldots, 1, n, n-1, \ldots, j+1, j$ where $2 \leq j - i \leq n-2$.

**Corollary**

Let $G$ be the $n$–cycle with the natural labeling of its vertices. Then

$$\text{in}_<(J_G) = (x_1y_2, \ldots, x_{n-1}y_n, x_1y_n, \{x_i x_{j+1} \cdots x_n y_1 \cdots y_{i-1} y_j\} \mid 2 \leq j - i \leq n-2).$$
$C_n, n \geq 4$

$\text{in}_<(J_G)$ is minimally generated by the initial monomials of the binomials corresponding to the edges of $G$ and by $m = n(n-3)/2$ monomials of degree $\geq 3$ which we denote by $v_1, \ldots, v_m$ where we assume that if $i < j$, then either $\deg v_i < \deg v_j$ or $\deg v_i = \deg v_j$ and $v_i > v_j$. 
In \( \text{in}_{<}(J_G) \) is minimally generated by the initial monomials of the binomials corresponding to the edges of \( G \) and by \( m = n(n-3)/2 \) monomials of degree \( \geq 3 \) which we denote by \( v_1, \ldots, v_m \) where we assume that if \( i < j \), then either \( \deg v_i < \deg v_j \) or \( \deg v_i = \deg v_j \) and \( v_i > v_j \).

If \( v_k = x_ix_{i+1} \cdots x_ny_1 \cdots y_{i-1}y_j \), we have \( \deg v_k = n-j+i+1 \).

Hence, there are two monomials of degree 3, namely, \( v_1 = x_1x_ny_{n-1} \) and \( v_2 = x_2y_1y_n \), three monomials of degree 4, namely, \( v_3 = x_1x_{n-1}x_ny_{n-2} \), \( v_4 = x_2x_ny_1y_{n-1} \), \( v_5 = x_3y_1y_2y_n \), etc.
Notation. \( J = (x_1y_2, x_2y_3, \ldots, x_{n-1}y_n) \), \( I = J + (x_1y_n) \), and, for \( 1 \leq k \leq m \), \( I_k = I_{k-1} + (v_k) \), with \( I_0 = I \). Therefore, \( I_m = \text{in}_<(J_G) \).
Notation. \( J = (x_1y_2, x_2y_3, \ldots, x_{n-1}y_n) \), \( I = J + (x_1y_n) \), and, for \( 1 \leq k \leq m \), \( l_k = l_{k-1} + (v_k) \), with \( l_0 = l \). Therefore, \( l_m = \text{in}_<(J_G) \).

**Lemma**

The ideals quotient \( J : (x_1y_n) \) and \( l_{k-1} : (v_k) \), for \( k \geq 1 \), are minimally generated by regular sequences of monomials of length \( n-1 \).
Theorem

Let $G$ be a cycle. Then $S/\text{in}_<(J_G)$ and $S/J_G$ have the same extremal Betti number, namely

$$\beta_{n,2n-2}(S/J_G) = \beta_{n,2n-2}(S/\text{in}_<(J_G)) = \binom{n-1}{2} - 1.$$
Matsuda and Murai \cite{MatsudaMurai}: If \( G \) is any connected graph on the vertex set \([n]\), we have

\[
\ell \leq \text{reg}(S/J_G) \leq n - 1,
\]

where \( \ell \) is the length of the longest induced path of \( G \).
Matsuda and Murai [Regularity bounds for binomial edge ideals, J. Commut. Algebra (2013)]: If $G$ is any connected graph on the vertex set $[n]$, we have

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where $\ell$ is the length of the longest induced path of $G$.


**Theorem**

Let $G$ be a closed graph. Then:

- $\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}(J_G)) = \text{the length of the longest induced path in } G$.  

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Example
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\]

**Question:** May we characterize the connected graphs \( G \) whose longest induced path has length \( \ell \) and \( \text{reg}(S/J_G) = \ell \)?
Example

\[ \text{reg}(S/J_G) = \text{reg}(S/\text{in}(J_G)) = 4 \]

**Question:** May we characterize the connected graphs $G$ whose longest induced path has length $\ell$ and $\text{reg}(S/J_G) = \ell$?

**Positive answer for trees** (almost never closed).
$C_\ell$-graphs

Let $G$ be a connected graph on the vertex set $[n]$ which consists of

(i) a sequence of maximal cliques $F_1, \ldots, F_\ell$ with $\dim F_i \geq 1$ for all $i$ such that $|F_i \cap F_{i+1}| = 1$ for $1 \leq i \leq \ell - 1$ and $F_i \cap F_j = \emptyset$ for any $i < j$ such that $j \neq i + 1$, together with

(ii) some additional edges of the form $F = \{j, k\}$ where $j$ is an intersection point of two consecutive cliques $F_i, F_{i+1}$ for some $1 \leq i \leq \ell - 1$, and $k$ is a vertex of degree 1.
Theorem

Let $G$ be a $C_\ell$-graph on the vertex set $[n]$. Then

$$\text{reg}(S/J_G) = \text{reg}(S/\text{in}_<(J_G)) = \ell$$

and

$$\text{depth } S/J_G = \text{depth } (S/\text{in}_<(J_G)) = n + 1.$$
Caterpillar trees

A *caterpillar tree* is a tree $T$ with the property that it contains a path $P$ such that any vertex of $T$ is either a vertex of $P$ or it is adjacent to a vertex of $P$. Clearly, any caterpillar tree is a $C_\ell$-graph for some positive integer $\ell$. 
Caterpillar trees

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Figure: Caterpillar
Let $T$ be a tree on the vertex set $[n]$ whose longest induced path $P$ has length $\ell$. Then $\text{reg}(S/J_T) = \ell$ if and only if $T$ is caterpillar.