On the resolution of binomial edge ideals

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The National School on Algebra

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Outline

- Objects

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- Problems and results

Combinatorial objects

Simple graphs: undirected, no loops, no multiple edges

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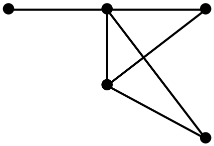

Algebraic objects

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[HHHKR] J. Herzog, T. Hibi, F. Hreinsdotir, T. Kahle, J. Rauh, **Binomial edge ideals and conditional independence statements**, Adv. Appl. Math. **45** (2010), 317–333.

M. Ohtani, **Graphs and Ideals generated by some 2-minors**, Commun. Algebra **39** (2011), no. 3, 905–917.

First examples

- $G = K_n$, $J_G = (f_{ij} : 1 \leq i < j \leq n) = I_2(X)$ where

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

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$$X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}.$$

- $G = L_n$, $J_G = (f_{i,i+1} : 1 \leq i \leq n-1)$, the ideal of adjacent minors of X.

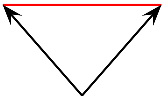
Quadratic Gröbner basis [HHHKR]

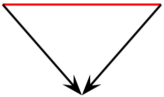
Theorem (HHHKR)

Let G be a simple graph on the vertex set $[n]$ with the edge set $E(G)$, and let $<$ be the lexicographic order on S induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Then the following conditions are equivalent:

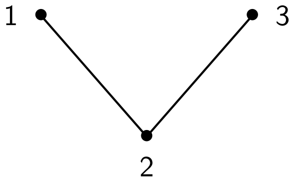
(a) *The generators f_{ij} of J_G form a quadratic Gröbner basis.*

(b) *For all edges $\{i,j\}$ and $\{i,k\}$ with $j > i < k$ or $j < i > k$ one has $\{j,k\} \in E(G)$.*





Closed graphs



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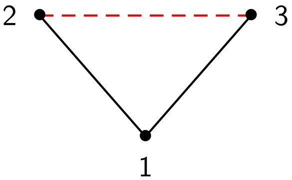

not closed

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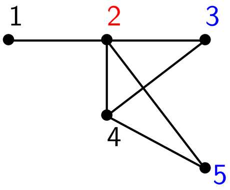

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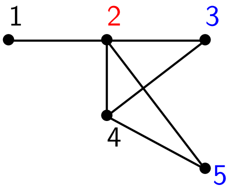

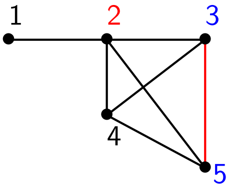

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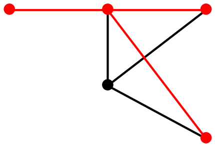

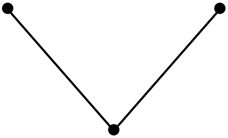

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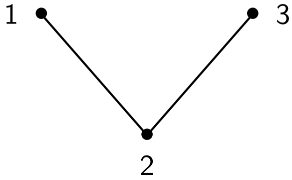

closed labeling

Closed graphs with CM binomial edge ideal: Betti numbers

Theorem (Ene, Herzog, Hibi)

Let G be a closed graph with Cohen–Macaulay binomial edge ideal. Then $\beta_{ij}(J_G) = \beta_{ij}(\mathrm{in}(J_G))$ for all i and j.

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Conjectures:

- $\beta_{ij}(J_G) = \beta_{ij}(\mathrm{in}_<(J_G))$ for all i and j for **any closed** graph G.

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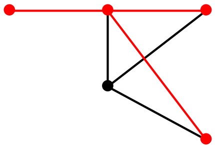
Conjectures:

- $\beta_{ij}(J_G) = \beta_{ij}(\mathrm{in}_<(J_G))$ for all i and j for **any closed** graph G.
- The **extremal Betti numbers of J_G and $\mathrm{in}_<(J_G)$ coincide for any graph G.**

Example



Example

J_G

	0	1	2	3	4
0 :	1	–	–	–	–
1 :	–	6	4	–	–
2 :	–	–	9	12	4
Total:	1	6	13	12	4

$\text{in}(J_G)$

	0	1	2	3	4
0 :	1	–	–	–	–
1 :	–	6	5	–	–
2 :	–	1	10	12	4
Total:	1	7	15	13	4

$K_{m,n}$

A. Dokuyucu, *Extremal Betti numbers of some classes of binomial edge ideals*, accepted, The Mathematical Reports.

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Betti diagram of S/J_G where $G = K_{m,n}$

	0	1	2	$\cdots$	p
0	1	0	0	$\cdots$	0
1	0	mn	0	$\cdots$	0
2	0	0	β_{24}	$\cdots$	$\beta_{p,p+2}$

where $p = \operatorname{proj\,dim} S/J_G = \begin{cases} m, & \text{if } n = 1, \\ 2m+n-2, & \text{if } n > 1. \end{cases}$

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S/J_G has a unique extremal Betti number, namely

$$\beta_{p,p+2} = \begin{cases} m-1, & \text{if } p = m, \\ n-1, & \text{if } p = 2m+n-2. \end{cases}$$

Gröbner basis [HHHKR]

Theorem (HHHKR)

The set of binomials

$$\mathscr{G} = \bigcup_{i<j} \{u_\pi f_{ij} : \pi \text{ is an admissible path from } i \text{ to } j\}$$

is the reduced Gröbner basis of J_G with respect to lexicographic order on S induced by the natural order of indeterminates, $x_1 > \cdots > x_n > y_1 > \cdots > y_n$.

Gröbner basis [HHHKR]

Definition

Let $i < j$ be two vertices of G. A path $i = i_0, i_1, \ldots, i_{r-1}, i_r = j$ from i to j is called **admissible** *if the following conditions are satisfied:*

(i) $i_k \neq i_\ell$ for $k \neq \ell$;

(ii) *for each $k = 1, \ldots, r-1$, one has either $i_k < i$ or $i_k > j$;*

(iii) *for any proper subset $\{j_1, \ldots, j_s\}$ of $\{i_1, \ldots, i_{r-1}\}$, the sequence $i, j_1, \ldots, j_s, j$ is not a path in G.*

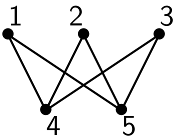
With a given admissible path π of G from i to j, we associate a monomial

$$u_\pi = (\prod_{i_k > j} x_{i_k})(\prod_{i_\ell < i} y_{i_\ell}),$$

where $\pi : i = i_0, i_1, \ldots, i_{r-1}, i_r = j$, from i to j, $i < j$.

Gröbner basis [HHHKR]

Let $G = K_{3,2}$ be a complete bipartite graph with 5 vertices.



The admissible paths of $K_{3.2}$ other than the edges are:

$$\pi_1 = 1,4,2;\ \ \pi_2 = 1,5,2;\ \ \pi_3 = 1,4,3;\ \ \pi_4 = 1,5,3;\ \ \pi_5 = 2,4,3;$$
$$\pi_6 = 2,5,3;\ \ \pi_7 = 4,1,5;\ \ \pi_8 = 4,2,5;\ \ \pi_9 = 4,3,5.$$

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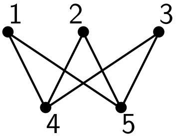

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$$\pi_6 = 2,5,3;\ \ \pi_7 = 4,1,5;\ \ \pi_8 = 4,2,5;\ \ \pi_9 = 4,3,5.$$

$$f_{14}, f_{15}, f_{24}, f_{25}, f_{34}, f_{35},$$

$$x_4 f_{12}, x_5 f_{12}, x_4 f_{13}, x_5 f_{13}, x_4 f_{23}, x_5 f_{23}, y_1 f_{45}, y_2 f_{45}, y_3 f_{45}$$

$K_{m,n}$

Corollary

Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $V(G) = \{1, \ldots, m\} \cup \{m+1, \ldots, m+n\}$. Then

$$\mathrm{in}_<(J_G) = (\{x_i y_j\}_{\substack{1\le i\le m \\ m+1\le j\le m+n}}, \{x_i x_{m+k} y_j\}_{\substack{1\le i<j\le m \\ 1\le k\le n}}, \{x_{m+i} y_k y_{m+j}\}_{\substack{1\le i<j\le n \\ 1\le k\le m}}).$$

$K_{m,n}$

Theorem

Let $G = K_{m,n}$ be the complete graph. Then
(a) $\mathrm{in}_<(J_G)$ *has linear quotients.*
(b) *The graded Betti numbers of* $\mathrm{in}_<(J_G)$ *are:*

$$\beta_{t,t+2}(\mathrm{in}_<(J_G)) = \sum_{\substack{1\le i\le m \\ m+1\le j\le m+n}} \binom{i+j-m-2}{t},$$

$$\beta_{t,t+3}(\mathrm{in}_<(J_G)) = \begin{cases} \sum_{\substack{1\le i<j\le m \\ 1\le k\le n}} \binom{n+k+j-3}{t}, & \text{if } n = 1, \\ \sum_{\substack{1\le i<j\le m \\ 1\le k\le n}} \binom{n+k+j-3}{t} + \sum_{\substack{1\le i<j\le n \\ 1\le k\le m}} \binom{m+k+j-3}{t}, & \text{if } n > 1. \end{cases}$$

$K_{m,n}$

Corollary

Let $G = K_{m,n}$ be the complete graph. Then:

(a) $\operatorname{proj\,dim}(S/\mathrm{in}_<(J_G)) = \operatorname{proj\,dim}(\mathrm{in}_<(J_G)) + 1 =$
$$\begin{cases} m, & \text{if } n = 1, \\ 2m+n-2, & \text{if } n > 1. \end{cases}$$

(b) $S/\mathrm{in}_<(J_G)$ has a unique extremal Betti number, namely

$$\beta_{p,p+2}(S/\mathrm{in}_<(J_G)) = \beta_{p-1,p+2}(\mathrm{in}_<(J_G)) = \begin{cases} m-1, & \text{if } n = 1, \\ n-1, & \text{if } n > 1. \end{cases}$$

$C_n, n \geq 4$

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1	0	n	0	0	$\cdots$	0
2	0	0	β_{24}	0	$\cdots$	0
3	0	0	0	β_{36}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	0	0	$\beta_{2,n}$	$\beta_{3,n+1}$	$\cdots$	$\beta_{n,2n-2}$

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$$\beta_{n,2n-2} = \binom{n-1}{2} - 1.$$

$C_n, n \geq 4$

Admissible paths in C_n : edges $i, i+1$ for $1 \leq i \leq n-1$ and $1, n$ together with $i, i-1, \ldots, 1, n, n-1, \ldots, j+1, j$ where $2 \leq j-i \leq n-2$.

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Corollary

Let G be the n–cycle with the natural labeling of its vertices. Then

$$\mathrm{in}_<(J_G) = (x_1y_2, \ldots, x_{n-1}y_n, x_1y_n, \{x_ix_{j+1}\cdots x_ny_1\cdots y_{i-1}y_j\}_{2\leq j-i\leq n-2}).$$

$C_n, n \geq 4$

$\mathrm{in}_<(J_G)$ is minimally generated by the initial monomials of the binomials corresponding to the edges of G and by $m = n(n-3)/2$ monomials of degree ≥ 3 which we denote by $v_1, \ldots, v_m$ where we assume that if $i < j$, then either $\deg v_i < \deg v_j$ or $\deg v_i = \deg v_j$ and $v_i > v_j$.

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If $v_k = x_i x_{j+1} \cdots x_n y_1 \cdots y_{i-1} y_j$, we have $\deg v_k = n - j + i + 1$.
Hence, there are two monomials of degree 3, namely,
$v_1 = x_1 x_n y_{n-1}$ and $v_2 = x_2 y_1 y_n$, three monomials of degree 4, namely, $v_3 = x_1 x_{n-1} x_n y_{n-2}, v_4 = x_2 x_n y_1 y_{n-1}, v_5 = x_3 y_1 y_2 y_n$, etc.

$C_n, n \geq 4$

Notation. $J = (x_1y_2, x_2y_3, \ldots, x_{n-1}y_n)$, $I = J + (x_1y_n)$, and, for $1 \leq k \leq m$, $I_k = I_{k-1} + (v_k)$, with $I_0 = I$. Therefore, $I_m = \text{in}_<(J_G)$.

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Lemma

The ideals quotient $J : (x_1y_n)$ and $I_{k-1} : (v_k)$, for $k \geq 1$, are minimally generated by regular sequences of monomials of length $n-1$.

$C_n, n \geq 4$

Theorem

Let G be a cycle. Then $S/\operatorname{in}_<(J_G)$ and S/J_G have the same extremal Betti number, namely
$\beta_{n,2n-2}(S/J_G) = \beta_{n,2n-2}(S/\operatorname{in}_<(J_G)) = \binom{n-1}{2} - 1.$

C_ℓ-graphs

Matsuda and Murai [*Regularity bounds for binomial edge ideals*, J. Commut. Algebra (2013)]: If G is any connected graph on the vertex set $[n]$, we have

$$\ell \leq \operatorname{reg}(S/J_G) \leq n-1,$$

where ℓ is the length of the longest induced path of G.

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Theorem

Let G be a closed graph. Then:

- $\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}(J_G)) =$ *the length of the longest induced path in G.*

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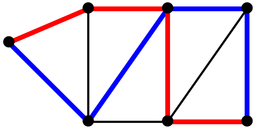
Example



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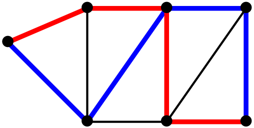

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}(J_G)) = 4$$

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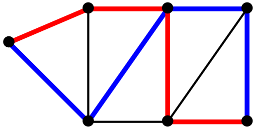
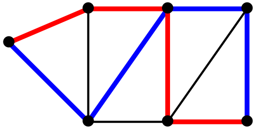


$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}(J_G)) = 4$$

Question: May we characterize the connected graphs G whose longest induced path has length ℓ and $\operatorname{reg}(S/J_G) = \ell$?

Example



$$\mathrm{reg}(S/J_G) = \mathrm{reg}(S/\mathrm{in}(J_G)) = 4$$

Question: May we characterize the connected graphs G whose longest induced path has length ℓ and $\mathrm{reg}(S/J_G) = \ell$?

Positive answer for trees (almost never closed).

C_ℓ-graphs

F. Chaudry, A. Dokuyucu, R. Irfan, *On the binomial edge ideals of block graphs,* submitted.

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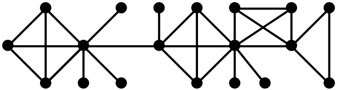

C_ℓ-graphs

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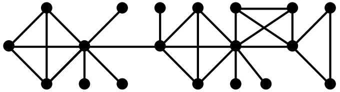

Let G be a connected graph on the vertex set $[n]$ which consists of

(i) a sequence of maximal cliques $F_1, \dots, F_\ell$ with $\dim F_i \geq 1$ for all i such that $|F_i \cap F_{i+1}| = 1$ for $1 \leq i \leq \ell - 1$ and $F_i \cap F_j = \emptyset$ for any $i < j$ such that $j \neq i + 1$, together with

(ii) some additional edges of the form $F = \{j, k\}$ where j is an intersection point of two consecutive cliques F_i, F_{i+1} for some $1 \leq i \leq \ell - 1$, and k is a vertex of degree 1.

C_ℓ-graphs

Theorem

Let G be a C_ℓ-graph on the vertex set $[n]$. Then

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}_<(J_G)) = \ell$$

and

$$\operatorname{depth} S/J_G) = \operatorname{depth}(S/\operatorname{in}_<(J_G)) = n+1.$$

Caterpillar trees

A *caterpillar tree* is a tree T with the property that it contains a path P such that any vertex of T is either a vertex of P or it is adjacent to a vertex of P. Clearly, any caterpillar tree is a C_ℓ-graph for some positive integer ℓ.

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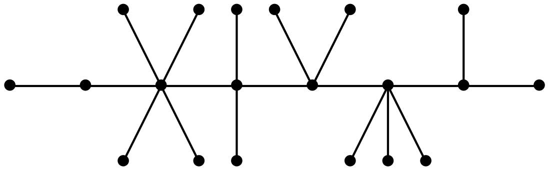

Figure: Caterpillar

Caterpillar

Theorem

Let T be a tree on the vertex set $[n]$ whose longest induced path P has length ℓ. Then $\mathrm{reg}(S/J_T) = \ell$ if and only if T is caterpillar.