SYZYGIES OF HIBI RINGS

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ABSTRACT. This is a very preliminary and extended version of my lectures in the school of algebra.

CONTENTS

INTRODUCTION

1. Hibi rings and their Gröbner bases

1.1. **Preliminaries of combinatorics.** In this section we review the definitions of the combinatorial objects that will be used throughout these lectures. For a comprehensive treatment and for references to the literature on this subject one may refer to the books of Stanley [31] and Birkoff [1].

Definition 1.1. A *partially ordered set* (*poset* in brief) is a set P endowed with a *partial order* $\leq$, that is, a relation which is

(i) reflexive: $x \leq x$ for all $x \in P$;
(ii) antisymmetric: for any $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$;
(iii) transitive: for any $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

We use the notation $x \geq y$ if $y \leq x$ and $x < y$ if $x \leq y$ and $x \neq y$. If $x \leq y$ or $y \leq x$ we say that x, y are *comparable* in P. Otherwise, x, y are *incomparable*.

All the posets in these lectures are assumed to be finite.

Examples 1.2. 1. Let $n \geq 1$ be an integer and $[n] = \{1, 2, \ldots, n\}$. This set is a poset with the natural order of integers. Any two elements of $[n]$ are comparable.
2. Let $B_n = 2^{[n]}$ be the power set of $[n]$. B_n is a partially ordered set with the inclusion. Obviously, not any two subsets of $[n]$ are comparable with respect to inclusion.
3. Let $n \geq 2$ be an integer and D_n the set of all divisors of n. D_n is partially ordered with respect to divisibility.

Any finite poset P is completely determined by its cover relations which are encoded in the Hasse diagram of P. We say that y *covers* x if $y > x$ and there is no $z \in P$ with $y > z > x$. In this case we write $y \gtrdot x$. The Hasse diagram of P is a graph whose vertices are the elements of P and the edges are the cover relations of P.

For example, in Figure 1, the Hasse diagrams of B_3 and D_{12} are displayed.
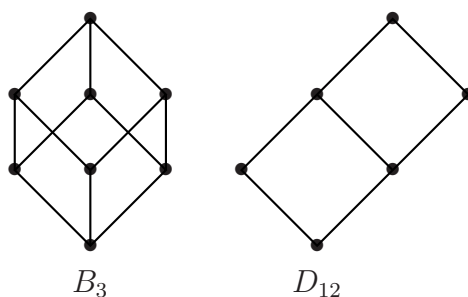

B_3 D_{12}

FIGURE 1

In Figure 2, we have the Hasse diagram of a poset P with 5 elements, x, y, z, t, u with $z \gtrdot x, z \gtrdot y, t \gtrdot y, t \gtrdot x, u \gtrdot t$.

A *subposet* of P is a subset Q endowed with a partial order such that, for $x, y \in Q$, we have $x \leq y$ in Q if and only if $x \leq y$ in P.
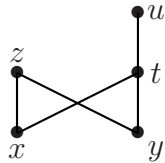
FIGURE 2. The Hasse diagram

For example, for the poset P displayed in Figure 2, the poset Q displayed in Figure 3 (a) is a subposet of P while the poset Q' displayed in Figure 3 (b) is not.
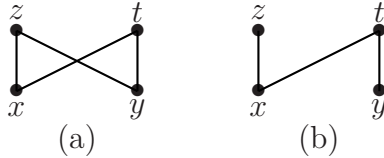

(a) (b)

FIGURE 3

Let P be a poset and $x, y \in P$ with $x \leq y$. The set

$$[x, y] = \{z \in P : x \leq z \leq y\}$$

is called a (*closed*) *interval* in P. Obviously, any interval of P is a subposet of P. For example, for the poset displayed in Figure 2, we have $[x, z] = \{x, z\}$ and $[y, u] = \{y, t, u\}$.

Definition 1.3. Let P and Q be two posets. An order preserving map $f : P \to Q$ is called a *morphism* of posets. The posets P and Q are called *isomorphic* if there exists a bijection $f : P \to Q$ which is a morphism of posets with the property that f^{-1} is a morphism as well.

A partial order on P is called a *total order* or *linear order* if any two elements of P are comparable, that is, for any $x, y \in P$, we have either $x \leq y$ or $y \leq x$. If $\leq$ is a total order on P, we call P a *totally ordered* set or *chain*. P is an *antichain* or *clutter* if any two different elements of P are incomparable.

Chain   Antichain

FIGURE 4

Given the poset P, a chain in P is a subposet C of P which is totally ordered. If C is a chain of P, $\ell(C) = |C| - 1$ is the *length* of C. A chain $C : x_0 < x_1 < \cdots < x_r$ in P is called *saturated* if $x_{i+1} \gtrdot x_i$ for all i.

Definition 1.4. Let P be a poset. The *rank* of P is

$$\text{rank}\, P = \max\{\ell(C) : C \text{ is a chain of } P\}.$$

If every maximal chain of P has the same length, then P is called *graded* or *pure*.

For example, the posets of Figure 1 are graded of rank 3 while the poset of Figure 2 is not graded.

A *minimal* element of a poset P is an element $x \in P$ such that, for any $y \in P$, if $y \leq x$ then $y = x$. In other words, if y, x are comparable, then $y \geq x$. By dualizing the above conditions, that is, taking $\geq$ instead of $\leq$, we define the *maximal* elements of P. For example, in the poset displayed in Figure 2 there are two minimal elements, namely x, y, and two maximal elements, z, u.

For a poset P, $\hat{P}$ denotes the poset $P \cup \{-\infty, \infty\}$ where, for $x, y \in P$, $x \leq y$ in $\hat{P}$ if and only if $x \leq y$ in P and $-\infty < x < \infty$ for all $x \in P$. For example, the poset $\hat{P}$ for the poset of Figure 2 is displayed in Figure 5.
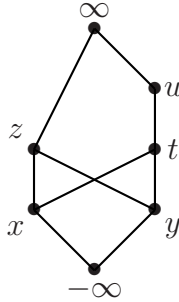

FIGURE 5. $\hat{P}$

Clearly, if $\hat{P}$ is graded if and only if P is graded.

For a graded poset of rank n, one considers the *rank function* $\rho : P \to \{0, 1, \ldots, n\}$ defined as follows:

$\rho(x) = 0$ for any minimal element of P;
$\rho(y) = \rho(x) + 1$ for $y > x$ in P.

If $\rho(x) = i$, we say that rank $x = i$.

Examples 1.5. 1. Let B_n be the Boolean poset on the set $[n]$. Then B_n is graded of rank n and, for $x \in B_n$, rank $x = |x|$.

2. Let $n \geq 2$ be an integer and D_n the poset of the divisors of n. The poset D_n is graded of rank equal to the number of the prime divisors of n and, for $x|n$, rank x is equal to the number of the prime divisor of x (in each case counted with multiplicity).

1.1.1. *Operations on posets.* **1. Direct sums.** Let P, Q be two posets on disjoint sets. The *direct sum* of P and Q is the poset $P + Q$ on the set $P \cup Q$ with the order defined as follows: $x \leq y$ in $P + Q$ if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q. A poset P which can be written as a direct sum of to subposets is called *disconnected*. Otherwise, P is *connected*.

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2. Ordinal sum. The *ordinal sum* $P \oplus Q$ of the disjoint posets P, Q is the poset on the set $P \cup Q$ with the following order. If $x, y \in P \oplus Q$, then $x \leq y$ if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q or $x \in P$ and $y \in Q$.

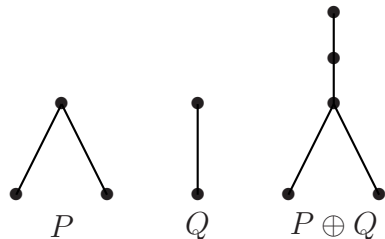**Example 1.6.** In Figure 6 the ordinal sum of two posets is displayed.

FIGURE 6. Ordinal sum

3. Cartesian product. Let P and Q be two posets. The cartesian product of P and Q is the poset $P \times Q$ on the set $P \times Q$ such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq x'$ in P and $y \leq y'$ in Q.

Example 1.7. Figure 7 shows a cartesian product of two posets.
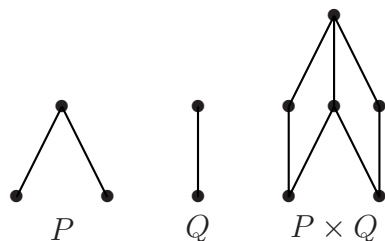
FIGURE 7. Cartesian product

4. The dual poset. Let P be a poset. The *dual* of P is the poset P^* on the same set as P such that $x \leq y$ in P^* if and only if $x \geq y$ in P. If P and P^* are isomorphic, then P is called *self-dual*.

Example 1.8. Figure 8 and Figure 9 show the dual of a poset and a self-dual poset.

1.1.2. *Lattices.* Let P be a poset and $x, y \in P$. An *upper bound* of x, y is an element $z \in P$ such that $z \geq x$ and $z \geq y$. If the set $\{z \in P : z$ is an upper bound of x and $y\}$ has a least element, this is obviously unique, is called the *join* of x and y, and it is denoted $x \vee y$. By duality, one defines the *meet* $x \wedge y$ of two elements x, y in a poset.

Definition 1.9. A lattice L is a poset with the property that for any $x, y \in L$, $x \vee y$ and $x \wedge y$ exist.

It is easily seen that if L and L' are lattices, then so are L^*, $L \oplus L'$, and $L \times L'$.
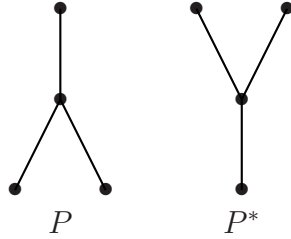
FIGURE 8. The dual poset
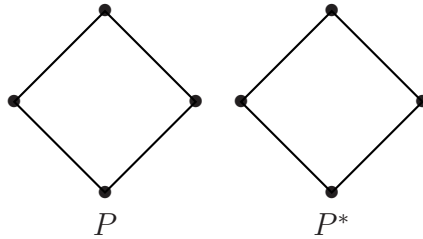
FIGURE 9. Self-dual poset

Example 1.10. B_n and D_n are lattices.

All the lattices considered in these lectures are finite. Unless otherwise stated, by a lattice we maen a finite lattice. Clearly, any lattice has a minimum and a maximum.

A sublattice of L is a subposet L' of L with the property that for any $x, y \in L'$, $x \vee y \in L'$ and $x \wedge y \in L'$.

Proposition 1.11. [31, Chapter 3] *Let L be a lattice. The following conditions are equivalent:*

(i) *L is graded and its rank function satisfies $\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$ for all $x, y \in L$.*

(ii) *If x and y cover $x \wedge y$, then $x \vee y$ covers x and y.*

Definition 1.12. A lattice L is called *modular* if it is graded and its rank function satisfies $\rho(x) + \rho(y) = \rho(x \wedge y) + \rho(x \vee y)$ for all $x, y \in L$.

The following proposition characterizes the modular lattices. For the proof one may consult [1].

Proposition 1.13. *Let L be a lattice. The following conditions are equivalent:*

(i) *L is a modular lattice;*

(ii) *For all $x, y, z \in L$ such that $x \leq z$, we have $x \vee (y \wedge z) = (x \vee y) \wedge z$.*

(iii) *L has no sublattice isomorphic to the pentagon lattice of Figure 10.*
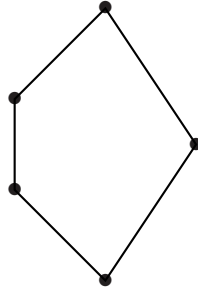
FIGURE 10. Pentagon

1.1.3. *Distributive lattices.* In these lectures we are mainly interested in distributive lattices.

Definition 1.14. Let L be a lattice. L is called *distributive* if satisfies one of the equivalent conditions:

    (i) for any $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
    (ii) for any $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

The lattices B_n and D_n are distributive while the lattices displayed in Figure 10 and Figure 11 are not.
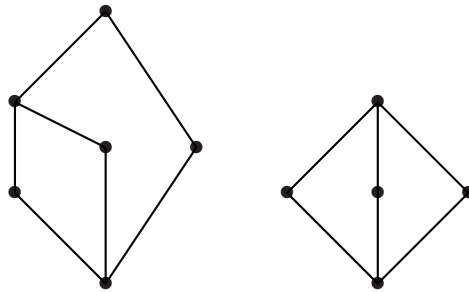
FIGURE 11. Non-distributive lattices

A famous theorem of Birkoff [1] states that every distributive lattice L is the lattice of the order ideals of a certain suposet P of L.

A subset α of a poset P is called an *order ideal* or *poset ideal* if it satisfies the following condition: for any $x \in \alpha$ and $y \in P$, if $y \leq x$, then $y \in \alpha$. The set of all order ideals of P is denoted $\mathcal{I}(P)$. The union and intersection of two order ideals are obviously order ideals. Therefore, $\mathcal{I}(P)$ is a distributive lattice with the union and intersection.

Given a lattice L, an element $x \in L$ is called *join-irreducible* if $x \neq \min L$ and whenever $x = y \vee z$ for some $y, z \in L$, we have either $x = y$ or $x = z$.

Theorem 1.15 (Birkoff)**.** *Let L be a distributive lattice and P its subposet of join-irreducible elements. Then L is isomorphic to $\mathcal{I}(P)$.*

In the following figure we illustrate Birkoff's theorem.

$$P \qquad\qquad L = \mathcal{I}(P)$$
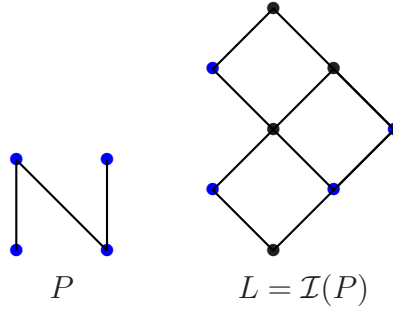
FIGURE 12. Birkoff's theorem

1.2. **Preliminaries of commutative algebra.** In this subsection we review basic facts about minimal free resolutions and canonical modules which will be needed in the sequel. We refer the reader to the book of Stanley [30], the survey of Herzog [15], and Chapter 4 in [7] for more information.

1.2.1. *Minimal graded free resolutions.* Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. The ring S is graded with the usual grading $S = \oplus_{d \geq 0} S_d$ where S_d is the K–vector space generated by all the monomials of S of degree d. The ideal $\mathfrak{m} = S_+ = \oplus_{d>0} S_d = (x_1, \ldots, x_n)$ is the unique graded maximal ideal of S.

A *graded S–module* M has a decomposition $M = \oplus_{n \in \mathbb{Z}} M_n$ as a vector space over K with the property that $S_d M_n \subset M_{n+d}$ for all d, n. Most often, we will work with graded modules of the form S/I where I is a graded ideal of S. A graded K–algebra of the form $R = S/I$ where I is a graded ideal of S is called a *standard graded algebra*.

All the graded S–modules considered in these lectures are finitely generated. Obviously, if M is a finitely generated graded S–module, then there exists $m \in \mathbb{Z}$ such that $M_n = 0$ for all $n < m$.

Let M be a graded S–module. The function $H(M, -) : \mathbb{Z} \to \mathbb{N}$ given by $H(M, n) = \dim_K M_n$, for all n, is called the *Hilbert function* of M. The generating series of this function, $H_M(t) = \sum_{n \in \mathbb{Z}} H(M, n) t^n$, is called the *Hilbert series* of M. For example, the Hilbert function of S is $H(S, d) = \binom{n+d-1}{n}$ and the Hilbert series is $H_S(t) = 1/(1-t)^n$.

If M is a graded S–module and a is an integer, then $M(a)$ is the graded module whose degree n component is $(M(a))_n = M_{a+n}$ for all n. By the definition of the Hilbert series, it obviously follows that $H_{M(a)}(t) = t^{-a} H_M(t)$.

Definition 1.16. A *graded free resolution* of the finitely generated graded S–module M is an exact sequence

$$F_\bullet : \qquad 0 \to F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

where F_i are free S–modules of finite rank and the maps $\varphi_i : F_i \to F_{i-1}$ preserve the degrees, that is, they are graded maps.

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The modules F_i are of the form $F_i = \oplus_{j \in \mathbb{Z}} S(-j)^{b_{ij}}$ for all i.

The resolution $\mathbb{F}_\bullet$ is called *minimal* if $\operatorname{Im} \varphi_i \subset \mathfrak{m} F_{i-1}$ for $i \geq 1$. This is equivalent to saying that all the matrices representing the maps φ_i in the resolution have all the entries in $\mathfrak{m}$. By the Hilbert's syzygy Theorem (see, for example, [7, Theorem 4.18]) it follows that $p \leq n$ if $\mathbb{F}$ is minimal.

Any two minimal graded free resolutions of M are isomorphic; see [7, Theorem 4.25]. Hence, if $\mathbb{F}_\bullet$ is minimal and $F_i = \oplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$, then the numbers $\beta_{ij} = \beta_{ij}(M)$ are called *the graded Betti numbers* of M. The numbers $\beta_i = \beta_i(M) = \sum_j \beta_{ij}$ are called *the total Betti numbers* of M. We have the following formulas for $\beta_i(M)$ and $\beta_{ij}(M)$:

$$\beta_i(M) = \dim_K \operatorname{Tor}_i^S(M, K) \text{ and } \beta_{ij}(M) = \dim_K \operatorname{Tor}_i^S(M, K)_j.$$

The following data can be red from the minimal graded free resolution of M. The *projective dimension* of M is defined as

$$\operatorname{proj dim} M = \max\{i : \beta_{ij \neq 0 \text{ for some } j}\}.$$

The *regularity* of M is given by

$$\operatorname{reg} M = \max\{j - i : \beta_{ij} \neq 0\}.$$

The graded Betti numbers of M are usually displayed in the so-called *Betti diagram* of M :

FIGURE 13. Betti diagram

Definition 1.17. Let M be a finitely generated graded S–module and $\{g_1, \ldots, g_m\}$ a minimal system of homogeneous generators of M. The module M has a *d–linear resolution* if $\deg g_i = d$ for $1 \leq i \leq m$ and $\beta_{ij}(M) = 0$ for $j \neq i + d$. In other words, we have $\beta_i(M) = \beta_{ii+d}(M)$ for all i.

Hence, M has a d–linear resolution if the minimal graded free resolution is of the form:

$$0 \to S(-d-p)^{\beta_p} \to \cdots \to S(-d-1)^{\beta_1} \to S(-d)^{\beta_0} \to M \to 0.$$

This is equivalent to saying that all the minimal homogeneous generators have degree d and all the maps in the minimal graded free resolution have linear form entries.
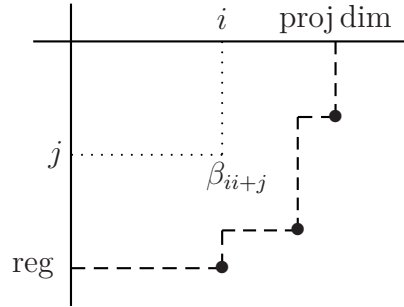
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Definition 1.18. Let M be a finitely generated graded S–module and $\{g_1, \ldots, g_m\}$ a minimal system of homogeneous generators of M. The module M has a *pure resolution* if its minimal graded free resolution is of the form

$$0 \to S(-d_p)^{\beta_p} \to \cdots \to S(-d_1)^{\beta_1} \to S(-d_0)^{\beta_0} \to M \to 0.$$

for some integers $0 < d_0 < d_1 < \cdots < d_p$.

1.2.2. *Cohen-Macaulay modules and canonical modules.*

Definition 1.19. Let M be a graded finitely generated S–module. A sequence of homogeneous elements $\theta_1, \ldots \theta_r \in \mathfrak{m}$ is called and M–*sequence* if θ_i is regular on $M/(\theta_1, \ldots, \theta_{i-1})M$ for all i which means that, for any i, the multiplication map $\theta_i : M/(\theta_1, \ldots, \theta_{i-1})M \to M/(\theta_1, \ldots, \theta_{i-1})M$ is injective.

The length of the longest M–sequence of homogeneous elements is called the *depth* of M. The Auslander-Buchsbaum Theorem [3, Theorem 1.3.3] states that

$$\operatorname{depth} M = n - \operatorname{proj\,dim} M.$$

In general, one has $\operatorname{depth} M \leq \dim M$; see [3, Proposition 1.2.12]. The equality case is very important in commutative algebra. A finitely generated graded S–module M is called *Cohen-Macaulay* if $\operatorname{depth} M = \dim M$.

Let $R = S/I$ be a standard graded Cohen-Macaulay algebra of dimension d with the minimal graded free S–resolution

$$\mathbb{F}_\bullet : \qquad 0 \to F_{n-d} \overset{\varphi_{n-d}}{\to} F_{n-d-1} \to \cdots \to F_1 \overset{\varphi_1}{\to} F_0 \overset{\varphi_0}{\to} R \to 0.$$

A finite graded S–module ω_R is the *canonical module* of R if

$$\operatorname{Ext}^i_S(S/\mathfrak{m}, \omega_R) \cong \begin{cases} 0, & i \neq d \\ S/\mathfrak{m}, & i = d \end{cases}.$$

Example 1.20. The Koszul complex of $x_1, \ldots, x_n$ gives the minimal graded free resolution of $K = S/\mathfrak{m}$. The last non-zero module in the Koszul complex is $F_n = S(-n)$. Then $\operatorname{Ext}^i_S(S/\mathfrak{m}, S) = 0$ for $i \neq n$ and $\operatorname{Ext}^n_S(S/\mathfrak{m}, S) = (S/\mathfrak{m})(n)$. Hence, the canonical module of S is $\omega_S = S(-n)$.

Let $R = S/I$ be a Cohen-Macaulay standard graded K–algebra and $\mathbb{F}_\bullet$ its minimal graded free resolution. Then the sequence

$$0 \to F_0^* \overset{\varphi_1^*}{\to} F_1^* \to \cdots \overset{\varphi_{n-d}^*}{\to} F_{n-d}^* \to \omega_R \to 0$$

is the minimal graded free resolution of ω_R. Therefore, we have

$$\beta_i(\omega_R) = \beta_{n-d-i}(R) \text{ for all } i.$$

In particular, $\beta_{n-d}(R)$ is equal to the minimal number of homogeneous generators of ω_R. The Betti number $\beta_{n-d}(R)$ is called the *type* of R and it is denoted $\operatorname{type}(R)$.

Definition 1.21. Let $R = S/I$ be a standard graded K–algebra. The algebra R is called *Gorenstein* if it is Cohen-Macaulay of type 1.

Hence, a Cohen-Macaulay standard graded K–algebra R is Gorenstein if and only if $\omega_R \cong R(a)$ for some integer a. The minimal free resolution of a Gorenstein algebra $R = S/I$ is self-dual. In particular, $\beta_i^S(R) = \beta_{n-d-i}^S(R)$.

Definition 1.22. Let $R = S/I$ be a standard graded Cohen-Macaulay K–algebra. The number
$$a(R) = -\min\{i : (\omega_R)_i \neq 0\}$$
is called the *a–invariant* of R.

Proposition 1.23. [3, Corollary 3.6.14] *Let $R = S/I$ be a standard graded Cohen-Macaulay K–algebra with canonical module ω_R and $\mathbf{y} = y_1, \ldots, y_m$ an R–sequence of homogeneous elements with $\deg y_i = a_i$ for $1 \leq i \leq m$. Then*

$$\omega_{R/\mathbf{y}R} \cong (\omega_R/\mathbf{y}\omega_R)(\sum_{i=1}^m a_i).$$

In particular, $a(R/\mathbf{y}R) = a(R) + \sum_{i=1}^m a_i$.

1.3. **Hibi rings and ideals.** In this subsection we describe a class of rings and binomial ideals which were introduced by Hibi in [19]. They are associated with finite distributive lattices.

Let L be a distributive lattice and $P = \{p_1, \ldots, p_n\}$ its set of join-irreducible elements. Thus, $L = \mathcal{I}(P)$. Let K be a field and $R = K[t, x_1, \ldots, x_n]$ be a polynomial ring in $n + 1$ indeterminates. Let $R[L]$ be the subring of R which is generated over K by the set of monomials $\{t \prod_{p_i \in \alpha} x_i : \alpha \in \mathcal{I}(P)\}$. Hibi showed in [19] that $R[L]$ is an algebra with straightening laws (ASL in brief) on P. We recall here the definition of an ASL. The reader may consult [5] for a quick introduction to this topic.

Let A be a K–algebra, H a finite poset, and $\varphi : H \to A$ an injective map. In the sequel we identify $x \in H$ with $\varphi(x) \in A$. A *standard monomial* in A is a monomial of the form $\alpha_1 \ldots \alpha_k$ where $\alpha_1 \leq \cdots \leq \alpha_k$.

Definition 1.24. A is called an ASL on H over K if the following hold:

(ASL-1) The set of standard monomials form a K–basis of A;

(ASL-2) If $\alpha, \beta \in H$ are incomparable and if $\alpha\beta = \sum r_i \gamma_{i1} \ldots \gamma_{ik_i}$ where $r_i \in K \setminus \{0\}$ and $\gamma_{i1} \leq \ldots \leq \gamma_{ik_i}$ is the unique expression of $\alpha\beta$ as a linear combination of standard monomials, then $\gamma_{i1} \leq \alpha\beta$ for all i.

The relations in axiom (ASL-2) are called the *straightening relations* of A. They generate the presentation ideal of A; see [5, Theorem 3.4].

Let $L = \mathcal{I}(P)$ be a distributive lattice with $P = \{p_1, \ldots, p_n\}$ and

$$\varphi : L \to R = K[t, x_1, \ldots, x_n]$$

given by

$$\varphi(\alpha) = t \prod_{p_i \in \alpha} x_i \text{ for } \alpha \in L.$$

One observes that, for any $\alpha, \beta \in L$,

(1) $$\varphi(\alpha)\varphi(\beta) = \varphi(\alpha \cap \beta)\varphi(\alpha \cup \beta).$$

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We show now that the Hibi ring $R[L]$ is an ASL on L over K. Axiom (ASL-2) is a straightforward consequence of equality (1).

For (ASL-1), it is enough to show that for any two chains $\alpha_1 \subseteq \cdots \subseteq \alpha_k$ and $\beta_1 \subseteq \cdots \subseteq \beta_\ell$ in L, we have $\varphi(\alpha_1) \ldots \varphi(\alpha_k) = \varphi(\beta_1) \ldots \varphi(\beta_\ell)$ if and only if $k = \ell$ and $\alpha_i = \beta_i$ for all i. This will imply that the standard monomials are distinct, so they form a K–basis of R.

Let $\varphi(\alpha_1) \cdots \varphi(\alpha_k) = \varphi(\beta_1) \cdots \varphi(\beta_\ell)$, that is, $t^k \prod_{i=1}^{k}(\prod_{p_j \in \alpha_i} x_j) = t^\ell \prod_{i=1}^{\ell}(\prod_{p_j \in \beta_i} x_j)$. This equality obviously implies that $k = \ell$ and $\prod_{i=1}^{k}(\prod_{p_j \in \alpha_i} x_j) = \prod_{i=1}^{k}(\prod_{p_j \in \beta_i} x_j)$. Therefore, we have

$$(\prod_{p_i \in \alpha_1} x_i)^k (\prod_{p_i \in \alpha_2 \setminus \alpha_1} x_i)^{k-1} \cdots (\prod_{p_i \in \alpha_k \setminus \alpha_{k-1}} x_i) = (\prod_{p_i \in \beta_1} x_i)^k (\prod_{p_i \in \beta_2 \setminus \beta_1} x_i)^{k-1} \cdots (\prod_{p_i \in \beta_k \setminus \beta_{k-1}} x_i).$$

This equality implies that $\alpha_i = \beta_i$ for $1 \le i \le k$.

Thus, we have proved that $R[L]$ is an ASL. Since the straightening relations generate the presentation ideal of $R[L]$, we get $R[L] \cong K[L]/I_L$ where $K[L] = K[\{x_\alpha : \alpha \in L\}]$ and $I_L = (x_\alpha x_\beta - x_{\alpha \cap \beta} x_{\alpha \cup \beta} : \alpha, \beta \in L, \alpha, \beta \text{ incomparable })$.

The presentation ideal I_L is called the (binomial[1]) *Hibi ideal* or the *join-meet ideal* of L.

1.3.1. *Gröbner bases of Hibi ideals.* As above, let $K[L]$ be the polynomial ring in the variables x_α with α in L and $I_L \subset K[L]$ the Hibi ideal associated with L. We order linearly the variables of $K[L]$ such that $x_\alpha \le x_\beta$ if $\alpha \subseteq \beta$. We consider the reverse lexicographic order $<$ on $K[L]$ induced by this order of the variables.

The following theorem appears, for instance, in [16]. We give here a different proof.

Theorem 1.25. [16, Theorem 10.1.3] *The generators of I_L form the reduced Gröbner basis of I_L with respect to $<$.*

Proof. For $\alpha, \beta \in L$, we set $f_{\alpha,\beta} = x_\alpha x_\beta - x_{\alpha \cap \beta} x_{\alpha \cup \beta}$. Obviously, $f_{\alpha,\beta} = 0$ if and only if α and β are comparable. If α, β are incomparable, then $\text{in}_<(f_{\alpha,\beta}) = x_\alpha x_\beta$.

According to Buchberger's criterion, it is enough to show that all the S–polynomials $S_<(f_{\alpha,\beta}, f_{\gamma,\delta})$ reduce to zero for any pair of generators $f_{\alpha,\beta}, f_{\gamma,\delta}$ of I_L. If $\text{in}_<(f_{\alpha,\beta})$ and $\text{in}_<(f_{\gamma,\delta})$ are relatively prime, then it is known that $S_<(f_{\alpha,\beta}, f_{\gamma,\delta})$ reduces to 0; see [7, Poposition 2.15]. It remains to show that any S–polynomial of the form $S_<(f_{\alpha,\beta}, f_{\alpha,\gamma})$ reduces to 0. But this follows immediately since one may easily check that the following equality is a standard expression of $S_<(f_{\alpha,\beta}, f_{\alpha,\gamma})$:

$$S_<(f_{\alpha,\beta}, f_{\alpha,\gamma}) = x_{\alpha \cup \gamma} f_{\beta,\alpha \cap \gamma} - x_{\alpha \cup \beta} f_{\gamma,\alpha \cap \beta} + x_{\alpha \cap \beta \cap \gamma}(f_{\alpha \cup \gamma, \beta \cup (\alpha \cap \gamma)} - f_{\alpha \cup \beta, \gamma \cup (\alpha \cap \beta)}).$$

$\square$

The above theorem has important consequences for the Hibi ring $R[L]$. In the first place, by Theorem 1.25, it follows that

$$\text{in}_<(I_L) = (x_\alpha x_\beta : \alpha, \beta \in L, \alpha, \beta \text{ incomparable}).$$

---

[1]One may find also the notion of monomial Hibi ideal in literature. But we do not discuss this notion in this lectures. Therefore, we will omit "binomial" when we refer to binomial Hibi ideals.

Hence, $\mathrm{in}_<(I_L)$ is a squarefree monomial ideal. A well-known theorem of Sturmfels [7, Theorem 5.16] implies that $R[L]$ is a normal domain. A theorem due to Hochster [21], shows that $R[L]$ is Chen-Macaulay.

The Cohen-Macaulay property of I_L may be deduced also in the following way. The initial ideal $\mathrm{in}_<(I_L)$ is the Stanley-Reisner ideal [2] of the order complex of L. This is defined as the complex of all the chains in L. By a theorem of Björner [3, Theorem 5.1.12], this complex is shellable, thus its Stanley-Reisner ideal is Cohen-Macaulay. Then I_L is Cohen-Macaulay as well; see [16, Corollary 3.3.5].

1.3.2. *Some comments.* We end this section with a few comments.

One may obviously consider the following more general settings. Let L be an arbitrary lattice, hence not necessarily distributive, and $K[L]$ the polynomial ring $K[\{x_\alpha : \alpha \in L\}]$. Let $I_L = (f_{\alpha,\beta} : \alpha, \beta \in L, \alpha, \beta \text{ incomparable})$. Hibi showed in [19] and it is easily seen that I_L is a prime ideal if and only if L is distributive.

One may naturally ask whether I_L is however a radical ideal when L is not distributive. This is not the case and one may check, for instance, that for the lattice L given in the left side of Figure 11, I_L is not a radical ideal. On the other hand, if L is a pentagon (Figure 10), then I_L is radical. The following problem would be of interest.

Problem 1.26. *Find classes of non-distributive lattices L with the property that I_L is a radical ideal.*

If I_L is a radical ideal, then its minimal prime ideals may be described in terms of the combinatorics of L; [12, Section 2].

A reverse lexicographic order $<$ in $K[L]$ with the property that $\mathrm{rank}\,\alpha < \mathrm{rank}\,\beta$ implies that $\alpha < \beta$ is called a *rank reverse lexicographic order*. The following theorem from [17] characterizes the distributive lattices amongst the modular lattices in terms of the Gröbner bases of their ideals.

Theorem 1.27. [17, Theorem 2.1] *Let L be a modular lattice. Then L is distributive if and only if I_L has a squarefree Gröbner basis with respect to any rank reverse lexicographic order.*

Moreover, in the same paper, the authors conjectured that if L is modular, then, for any monomial order, $\mathrm{in}_<(I_L)$ is not sqaurefree, unless L is distributive. This conjecture was proved in [12]:

Theorem 1.28. *Let L be a modular non-distributive lattice. Then, for any monomial order $<$ on $K[L]$, the initial ideal $\mathrm{in}_<(I_L)$ is not squarefree.*

Note that, for the diamond lattice L (the lattice displayed in Figure 11 in the right side), I_L is radical. However, as L is modular, none of its initial ideals $\mathrm{in}_<(I_L)$ is squarefree. This simple example shows that the approach of Problem 1.26 is not so easy. One of the most common techniques to show that a polynomial ideal I is radical is to find an initial ideal of I which is radical. Unfortunately, as we have seen in Theorem 1.28, this technique cannot be applied in approaching Problem 1.26.

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[2]For the Stanly-Reisner theory we refer the reader to the monographs [3, 30].

1.4. **The canonical module of a Hibi ring.** Let $L = \mathcal{I}(P)$ be a distributive lattice with $P = \{p_1, \ldots, p_n\}$ and $R[L] \subset K[t, x_1, \ldots, x_n]$ the associated Hibi ring. As we have already seen, $R[L]$ is and ASL on L over K which has as K–basis the standard monomials. This implies that every monomial in $R[L]$ is of the form $t^{w_0} x_1^{w_1} \cdots x_n^{w_n}$ where $(w_0, w_1, \ldots, w_n) \in \mathbb{N}^{n+1}$ with $w_0 \geq w_i$ for all i and $w_i \geq w_j$ if $p_i \leq p_j$ in P.

Since $R[L]$ is a domain, the canonical module ω_L of $R[L]$ is an ideal of $R[L]$; see [3, Proposition 3.3.18]. By a theorem of Stanley [29], a K–basis of the canonical ideal ω_L is given by the monomials $t^{w_0} x_1^{w_1} \cdots x_n^{w_n} \in R[L]$ with $w_0 > w_i > 0$ for all i and $w_i > w_j$ if $p_i < p_j$ in P.

Let $\hat{P} = P \cup \{-\infty, \infty\}$ be the poset defined in Subsection 1.1 and $\mathcal{S}(P)$ the set of all functions $v : \hat{P} \to \mathbb{N}$ with $v(\infty) = 0$ and $v(p) \leq v(q)$ if $p \geq q$ in $\hat{P}$. A function v as above is called an *order reversing map*. A function $v \in \mathcal{S}(P)$ is a *strictly order reversing map* if $v(p) < v(q)$ if $p > q$ in $\hat{P}$. Let $\mathcal{T}(P)$ be the set of all strictly order reversing maps on $\hat{P}$. Then, from what we said above, it follows that a K–basis of the canonical ideal ω_L is given by the set $\{v^{(-\infty)} \prod_{i=1}^n x_i^{v(p_i)} : v \in \mathcal{T}(P)\}$.

On $\mathcal{T}(P)$ one defines the following partial order. For $v, v' \in \mathcal{T}(P)$, we set $v \geq v'$ if the following conditions hold:

(i) $v(p) \geq v'(p)$ for all $p \in \hat{P}$,

(ii) the function $v - v' \in \mathcal{S}(P)$, where $v - v' : \hat{P} \to \mathbb{N}$ is defined by $(v - v')(p) = v(p) - v'(p)$ for all $p \in \hat{P}$.

It follows that the minimal generators of ω_L are in one-to-one correspondence with the minimal elements of the poset $\mathcal{T}(P)$. In particular, $R[L]$ is Gorenstein if and only if $\mathcal{T}(P)$ has a unique minimal element.

The following theorem was proved in [19, §3]. Before stating it, we need to introduce some notation. For $x \in \hat{P}$, depth x denotes the rank of the subposet of $\hat{P}$ consisting of all elements $y \geq x$ in $\hat{P}$, and height x denotes the rank of the subposet of $\hat{P}$ which consists of all $y \in \hat{P}$ with $y \leq x$. The number coheight $x =$ rank $\hat{P}$ − height x is called the *coheight* of x. It is clear that the functions depth and coheight belong to $\mathcal{T}(P)$. In addition, on easily sees that, for any $x, y \in \hat{P}$ with $x > y$, we have depth $y \geq$ depth $x + 1$ and height $x \geq$ height $y + 1$. If P is pure, then depth $x +$ height $x =$ rank $\hat{P}$ for any $x \in \hat{P}$.

Let $v \in \mathcal{T}(P)$ and $-\infty < p_0 < p_1 < \cdots < p_r < \infty$ be a maximal chain in $\hat{P}$ with $r =$ rank P. Then $v(-\infty) > v(p_0) > v(p_1) > \cdots > v(P_r) > v(\infty) = 0$, which implies that

(2) $$v(-\infty) \geq \operatorname{rank} \hat{P} = \operatorname{rank} P + 2.$$

with similar arguments, one shows that, for all $x \in \hat{P}$,

(3) $$v(x) \geq \operatorname{depth} x.$$

Theorem 1.29. [19] *The Hibi ring $R[L]$ is Gorenstein if and only if P is pure.*

Proof. To begin with, let P be pure and $x, y \in \hat{P}$ with $x > y$. We get

$$\operatorname{height} y + \operatorname{depth} x + 1 = \operatorname{rank} \hat{P} = \operatorname{height} y + \operatorname{depth} y.$$

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This implies that depth $y =$ depth $x + 1$. By using this equality, we show that depth is the unique minimal element of $\mathcal{T}(P)$. Let $v \in \mathcal{T}(P)$. Obviously, $v(x) \geq$ depth x for all $x \in \hat{P}$. Let now $x, y \in \hat{P}$ with $x \gtrdot y$. Then $v(x) -$ depth $x \leq v(y) -$ depth y since $v(x) - v(y) \leq -1 =$ depth $x -$ depth y. Clearly, the inequality $v(x) -$ depth $x \leq v(y) -$ depth y extends to any $x > y$ in $\hat{P}$ which shows that $v \geq$ depth in $\mathcal{T}(P)$.

Conversely, let $R[L]$ be a Gorenstein ring, that is, $\mathcal{T}(P)$ has a unique minimal element. Assume that P is not pure. Then there must be $x, y \in \hat{P}$ with $x \gtrdot y$ such that depth $y >$ depth $x + 1$. We define $w \in \mathcal{T}(P)$ as follows,

$$w(z) = \begin{cases} 1 + \text{depth } z, & z \leq x, z \neq y, \\ \text{depth } z, & \text{otherwise.} \end{cases}$$

Then $w(z) \geq$ depth z for all z and $w(x) -$ depth $x = 1 > w(y) -$ depth $y = 0$. This shows that w and depth are incomparable in $\mathcal{T}(P)$ which implies that $\mathcal{T}(P)$ has at least two minimal elements, a contradiction to our hypothesis. $\qquad\square$

Examples 1.30.   1. For the lattice L displayed in Figure 12, the ring $R[L]$ is Gorenstein since the poset of the join-irreducible elements is pure.
2. Let $P = \{p_1, p_2, p_3\}$ with $p_1 < p_2$. This poset is not pure, thus the Hibi ring of the lattice $L = \mathcal{I}(P)$ is not Gorenstein.

1.5. Generalized Hibi rings. Hibi rings were generalized in [11]. Let $P = \{p_1, \ldots, p_n\}$ be a poset and $\mathcal{I}(P)$ the ideal lattice of P. We fix a positive integer r. An r–*multichain* in P is a chain of poset ideals of P of length r:

$$\mathcal{I}: \qquad I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{r-1} \subseteq I_r = P.$$

Let $\mathcal{I}_r(P)$ be the set of all r–multichains in P. If $\mathcal{I} : I_1 \subseteq \cdots \subseteq I_{r-1} \subseteq I_r = P$ and $\mathcal{J} : J_1 \subseteq \cdots \subseteq J_{r-1} \subseteq J_r = P$ are two r–multichains in P, then

$$\mathcal{I} \cup \mathcal{J} : I_1 \cup J_1 \subseteq \cdots \subseteq I_{r-1} \cup J_{r-1} \subseteq I_r \cup J_r = P$$

and

$$\mathcal{I} \cap \mathcal{J} : I_1 \cap J_1 \subseteq \cdots \subseteq I_{r-1} \cap J_{r-1} \subseteq I_r \cap J_r = P$$

belong to $\mathcal{I}_r(P)$ as well, hence $\mathcal{I}_r(P)$ is a distributive lattice.

With each r–multichain $\mathcal{I}$ in $\mathcal{I}_r(P)$ we associate a monomial $u_\mathcal{I}$ in the polynomial ring $S = K[\{x_{ij} : 1 \leq i \leq r, 1 \leq j \leq n\}]$ which is defined as

$$u_\mathcal{I} = x_{1J_1} x_{2J_2} \cdots x_{rJ_r}$$

where

$$x_{kJ_k} = \prod_{p_\ell \in J_k} x_{k\ell} \text{ and } J_k = I_k \setminus I_{k-1} \text{ for all } 1 \leq k \leq r.$$

Let $R_r(P)$ be the K–subalgebra of S generated by the set $\{u_\mathcal{I} : \mathcal{I} \in \mathcal{I}_r(P)\}$. The ring $R_r(P)$ is called a *generalized Hibi ring*.

For example, for $r = 2$, an r–multichain of P is of the form $I \subseteq P$ where I is a poset ideal of P. If we set $x_{1j} = x_j$ and $x_{2j} = y_j$ for $1 \leq j \leq n$, then to the multichain $I \subset P$ we associate the monomial $\prod_{p_i \in I} x_i \prod_{p_i \notin I} y_i$. Hence,

$$R_2(P) = K[\{\prod_{p_i \in I} x_i \prod_{p_i \notin I} y_i : I \in \mathcal{I}(P)\}].$$

The ring $R_2(P)$ is isomorphic to the classical Hibi ring associated with the lattice $L = \mathcal{I}(P)$ since they have the same defining relations as it follows as a particular case of Corollary 1.32.

Similarly to the classical Hibi rings, we get the following result.

Theorem 1.31. *The ring $R_r(P)$ is an ASL on $\mathcal{I}_r(P)$ over K.*

Proof. The proof is similar to the corresponding statement for Hibi rings.

Let $\psi : \mathcal{I}_r(P) \to S$ defined by $\psi(\mathcal{I}) = u_\mathcal{I}$ for all $\mathcal{I} \in \mathcal{I}_r(P)$. One may check that

$$\psi(\mathcal{I})\psi(\mathcal{J}) = \psi(\mathcal{I} \cap \mathcal{J})\psi(\mathcal{I} \cup \mathcal{J})$$

for all $\mathcal{I}, \mathcal{J} \in \mathcal{I}_r(P)$; see also [11, Lemma 2.1]. This equality shows that $R_r(P)$ satisfies axiom (ASL-2). For showing (ASL-1), one may proceed as in Subsection 1.3 and show that the standard monomials in $R_r(P)$ are distinct. Indeed, let $\mathcal{I}_1 \subseteq \cdots \subseteq \mathcal{I}_t$ and $\mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_s$ be two chains in $\mathcal{I}_r(P)$ such that

$$\psi(\mathcal{I}_1) \cdots \psi(\mathcal{I}_t) = \psi(\mathcal{J}_1) \cdots \psi(\mathcal{J}_s)$$

which is equivalent to

$$\prod_{q=1}^{t} u_{\mathcal{I}_q} = \prod_{q=1}^{s} u_{\mathcal{J}_q}$$

or, more explicitly,

$$\prod_{q=1}^{t}(\prod_{k=1}^{r} x_{k,\mathcal{I}_{q,k}\backslash\mathcal{I}_{q,k-1}}) = \prod_{q=1}^{s}(\prod_{k=1}^{r} x_{k,\mathcal{J}_{q,k}\backslash\mathcal{J}_{q,k-1}}).$$

From this last equality it follows

$$\prod_{q=1}^{t}(\prod_{k=1}^{\ell} x_{k,\mathcal{I}_{q,k}\backslash\mathcal{I}_{q,k-1}}) = \prod_{q=1}^{s}(\prod_{k=1}^{\ell} x_{k,\mathcal{J}_{q,k}\backslash\mathcal{J}_{q,k-1}})$$

for all $\ell \in \{1, \ldots, r\}$. Taking $\ell = 1$ in the above equality we derive $t = s$ and $I_{q1} = J_{q1}$. Next, by inspecting the above equalities step by step for $\ell = 2, \ldots, r$, we get $\mathcal{I}_{q,k} = \mathcal{J}_{q,k}$ for all q and k. $\square$

Let T be the polynomial ring in the variables $y_\mathcal{I}$ with $\mathcal{I} \in \mathcal{I}_r(P)$ and $\varphi : T \to R_r(P)$ the K–algebra homomorphism induced by $y_\mathcal{I} \mapsto u_\mathcal{I}$ for all $\mathcal{I} \in \mathcal{I}_r(P)$. Theorem 1.31 has the following consequence.

Corollary 1.32. *The presentation ideal of the ring $R_r(P)$ is generated by the binomials $y_\mathcal{I} y_\mathcal{J} - y_{\mathcal{I} \cap \mathcal{J}} y_{\mathcal{I} \cup \mathcal{J}}$ where $\mathcal{I}, \mathcal{J} \in \mathcal{I}_r(P)$ are incomparble r–multichains.*

We fix a linear order on the variables $y_\mathcal{I}$ such that $y_\mathcal{I} < y_\mathcal{J}$ if $\mathcal{I} \subset \mathcal{J}$. Corollary 1.32 shows that $R_r(P)$ is the classical Hibi ring of $\mathcal{I}_r(P)$, thus we get the following statement.

Theorem 1.33. [11, Theorem 4.1] *The set*

$$\mathcal{G} = \{y_\mathcal{I} y_{\mathcal{I}'} - y_{\mathcal{I} \cup \mathcal{I}'} y_{\mathcal{I} \cap \mathcal{I}'} \in T : \mathcal{I}, \mathcal{I}' \in \mathcal{I}_r(P) \text{ incomparable}\},$$

is the reduced Gröbner basis of the ideal $\text{Ker } \varphi$ with respect to the reverse lexicographic order induced by the given order of the variables $y_\mathcal{I}$.

Corollary 1.34. [11, Corollary 4.2] *The ring $R_r(P)$ is a Cohen-Macaulay normal domain.*

In order to have a better knowledge of $R_r(P)$, we need to identify the join-irreducible elements of $\mathcal{I}_r(P)$. Let Q_{r-1} denote the set $[r-1] = \{1, \ldots, r-1\}$ endowed with the natural order.

Theorem 1.35. [11, Theorem 4.3] *Let P be a finite poset. Then, for any $r \geq 2$, $R_r(P) \cong R_2(P \times Q_{r-1})$.*

Proof. We have to show that the poset P' of the join-irreducible elements of $\mathcal{I}_r(P)$ is isomorphic to $P \times Q_{r-1}$.

In the first place we identify the join-irreducible elements of P'. Let

$$\mathcal{I} : \emptyset \subseteq \emptyset \subseteq \cdots \subseteq \emptyset \subset I_k \subseteq \cdots \subseteq I_r = P$$

be an r–multichain of $\mathcal{I}_r(P)$. We claim that $\mathcal{I}$ is join-irreducible if and only if I_k is a join-irreducible poset ideal in P and $I_k = I_{k+1} = \cdots = I_{r-1}$. The if part is obvious. For the only if part, let us first assume that $I_k = J \cup J'$ with J, J' poset ideals different from I_k. Then, we may decompose $\mathcal{I} = \mathcal{J} \cup \mathcal{J}'$ where

$$\mathcal{J} : \emptyset \subseteq \emptyset \subseteq \cdots \subseteq \emptyset \subset J \subseteq \cdots \subseteq I_r = P$$

and

$$\mathcal{J}' : \emptyset \subseteq \emptyset \subseteq \cdots \subseteq \emptyset \subset J' \subseteq \cdots \subseteq I_r = P,$$

a contradiction.

Suppose now that there exists an integer s with $k \leq s < r - 1$ such that $I_k = I_{k+1} = \cdots = I_s$ and $I_s \subset I_{s+1}$. Then $\mathcal{I} = \mathcal{J} \cup \mathcal{J}'$ where

$$\mathcal{J} : \emptyset \subseteq \cdots \subseteq \emptyset \subset I_k \subseteq I_k \subseteq \cdots \subseteq I_k \subseteq I_r$$

and

$$\mathcal{J}' : \emptyset \subseteq \cdots \subseteq \emptyset \subset I_s \subseteq I_{s+1} \subseteq \cdots \subseteq I_r,$$

a contradiction.

Let $\mathcal{I} : \emptyset \subseteq \cdots \subseteq \emptyset \subset I = I = \cdots = I \subset P$ with k copies of I where I is a join irreducible element of $\mathcal{I}(P)$. Then I is a principal ideal in $\mathcal{I}(P)$, hence there exists a unique element $p \in I$ such that $I = \{a \in P : a \leq p\}$. We define the poset isomorphism between the poset P' of the join irreducible elements of $\mathcal{I}_r(P)$ and $P \times Q_{r-1}$ as follows. To $\mathcal{I} : \emptyset \subseteq \cdots \subseteq \emptyset \subset I = I = \cdots = I \subset P$ with k copies of I we assign $(p, k) \in P \times Q_{r-1}$. $\qquad \square$

The above theorem allows us to extend Theorem 1.29 to generalized Hibi rings.

Corollary 1.36. [11, Corollary 4.5] *Let $r \geq 2$ be an integer. The ring $R_r(P)$ is Gorenstein if and only if P is pure.*

Proof. By Theorem 1.35 and Theorem 1.29, we only need to observe that the poset P' of the join irreducible elements of $\mathcal{I}_r(P)$ is pure if and only if P is pure. $\qquad \square$

2. Level and pseudo-Gorenstein Hibi rings

In Theorem 1.29 we presented the characterization of Gorenstein Hibi rings in terms of the poset P of the join-irreducible elements of the lattice. In this section we study two weaker properties of $R[L]$. More precisely, we will characterize the Hibi rings which are level or pseudo-Gorenstein. This section is mainly based on paper [9].

2.1. Level and pseudo-Gorenstein algebras.

Let K be a field and R a standard graded K–algebra. We assume that R has the presentation $R = S/I$ where $S = K[x_1, \ldots, x_n]$ is a polynomial ring over K and $I \subset S$ is a graded ideal. We also make the assumption that R is Cohen-Macaulay of dimension d. Let ω_R denote the canonical module of R and $a = \min\{i : (\omega_R)_i \neq 0\}$.

As we have already seen in Subsection 1.2.2, R is Gorenstein if and only if ω_R is a cyclic R–module. Let

$$\mathbb{F} : \quad 0 \to F_{n-d} \to \cdots \to F_1 \to F_0 \to R \to 0$$

be the minimal graded free resolution of R over S.

The notion of level rings was introduced in [28].

Definition 2.1. The algebra R is called *level* if all the generators of the canonical module ω_R have the same degree.

In other words, R is level if and only if the generators of F_{n-d} are of same degree. The following notion was introduced in [9].

Definition 2.2. The algebra R is called *pseudo-Gorenstein* if $\dim_K(\omega_R)_a = 1$.

It is already clear from the above definitions that an algebra R is Gorenstein if it is level and pseudo-Gorenstein.

On the other hand, we may easily prove the following characterization of pseudo-Gorensteiness.

Proposition 2.3. *Let R be a Cohen-Macaulay standard graded K–algebra of $\dim R = d$ and canonical module ω_R. The following statements are equivalent:*

(i) *The algebra R is pseudo-Gorenstein;*
(ii) *Let $\mathbf{y} = y_1, \ldots, y_d$ be an R–regular sequence of linear forms and $\bar{R} = R/\mathbf{y}R$. Let $b = \max\{i : (\bar{R})_i \neq 0\}$. Then $\dim_K(\bar{R})_b = 1$.*
(iii) *Let $H_R(t) = h(t)/(1-t)^d$ be the Hilbert series of R. Then, the leading coefficient of h is equal to 1.*
(iv) *The highest shift c in the resolution $\mathbb{F}$ of R over S appears in F_{n-d} and $\beta_{n-d,c}(R) = 1$.*

Proof. We briefly sketch the main steps of the proof. Implication (iv)$\Rightarrow$(iii) follows immediately if we apply the additivity property of the Hilbert series to the resolution of R. We get

$$H_R(t) = \frac{\sum_{i=0}^{n-d}(-1)^i \sum_j \beta_{ij}t^j}{(1-t)^n}.$$

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The leading coefficient of the numerator of $H_R(t)$ is equal to $(-1)^{n-d}$, hence after simplifying the expression of $H_R(t)$ by $(1-t)^{n-d}$ we get the leading coefficient of $h(t)$ equal to 1.

For (iii)$\Rightarrow$(ii), we notice that $H_R(t) = H_{\bar{R}}(t)/(1-t)^d$, thus $H_{\bar{R}}(t) = \sum_{i=0}^{b} \dim_K(\bar{R})_i t^i = h(t)$. This equality leads to the desired conclusion.

Implication (ii)$\Rightarrow$(i) follows by Proposition 1.23 combined with the fact that the canonical module of $\bar{R}$ is $\operatorname{Hom}_K(\bar{R}, K)$ since $\bar{R}$ is Artinian; see [3, Theorem 3.3.7].

Finally, (i)$\Rightarrow$(ii) is obvious since the resolution of ω_R is the dual of $\mathbb{F}$. $\qquad\square$

In the next two subsections, we will study level and pseudo-Gorenstein Hibi rings. It will turn out that the property of $R[L]$ of being pseudo–Gorenstein or level does not depend on the field. Therefore, we may also say that L is pseudo–Gorenstein or level if the Hibi ring is so.

2.2. Pseudo-Gorenstein Hibi rings.
Let L be a distributive lattice and P the subposet of its join-irreducible elements. Let $S = K[\{x_\alpha : \alpha \in L\}]$ and I_L the Hibi binomial ideal associated with L. As we have seen in Subsection 1.4, the canonical ideal ω_L of $R[L] = S/I_L$ has the minimal generators in one-to-one correspondence with the minimal elements of the poset $\mathcal{T}(P)$ which consists of all strictly order reversing maps $v : \hat{P} \to \mathbb{N}$. It then follows that $R[L]$ is pseudo-Gorenstein if and only if $\mathcal{T}(P)$ contains exactly one minimal element.

Theorem 2.4. [9, Theorem 2.1][31, Corollary 3.15.18 (a)] *The ring $R[L]$ is pseudo-Gorenstein if and only if, for all $x \in P$, we have $\operatorname{depth}(x) + \operatorname{height}(x) = \operatorname{rank} \hat{P}$.*

Proof. Let $R[L]$ be pseudo-Gorenstein. Then ω_L has a unique minimal generator of least degree which is actually $\operatorname{rank} \hat{P}$. Since the maps depth and coheight correspond to generators of degree equal to $\operatorname{rank} \hat{P}$, they must be equal. This leads to the desired equality.

Conversely, let us assume that for all $x \in P$, we have $\operatorname{depth}(x) + \operatorname{height}(x) = \operatorname{rank} \hat{P}$. This implies that, for any $x \in \hat{P}$, there exists a chain C of length equal to $\operatorname{rank} \hat{P}$ with $x \in C$. Let $v \in \mathcal{T}(P)$ with $v(-\infty) = \operatorname{rank} \hat{P}$. Then, for any $y \in C$, we must have $v(y) = \operatorname{depth}(y)$. In particular, $v(x) = \operatorname{depth}(x)$. Hence, v is uniquely determined which implies that L is pseudo-Gorenstein. $\qquad\square$

In Figure 14 we represent the posets P for a pseudo-Gorenstein lattice which is not Gorenstein and a lattice which is not pseudo-Gorenstein.
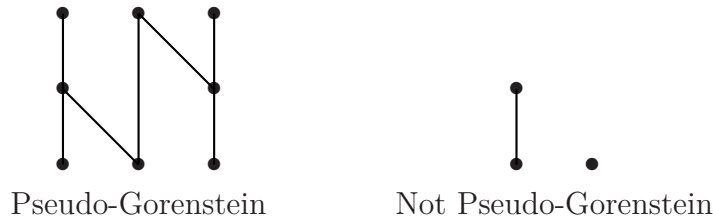

Pseudo-Gorenstein   Not Pseudo-Gorenstein

FIGURE 14

2.3. **Level Hibi rings.** The examples displayed in Figure 15 are taken from [20]. They show that it is not possible to decide the levelness of a Hibi ring only from its h–vector.

$$h(R[L]) = (1, 7, 9, 2)$$
$$R[L] \text{ is level}$$

$$h(R[L]) = (1, 6, 9, 2)$$
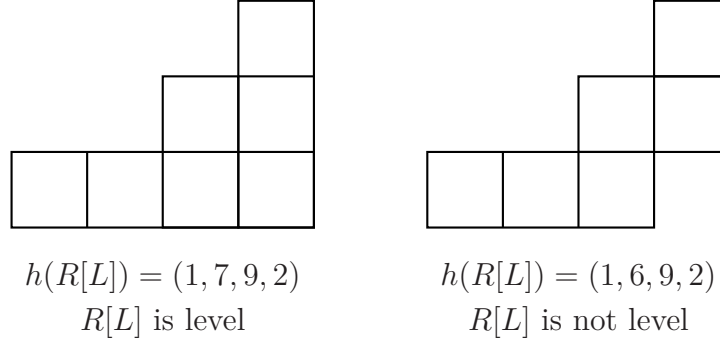$$R[L] \text{ is not level}$$

FIGURE 15

The first attempt to study the level property of a Hibi ring was done in [23]. In that paper, a sufficient condition for levelness was given.

Theorem 2.5. [23, Theorem 3.3] *Let $L = \mathcal{I}(P)$ be a distributive lattice. If the subposet $\{y \in P : y \geq x\}$ of P is pure for all $x \in P$, then $R[L]$ is level.*

Proof. We have to prove that all the minimal elements of $\mathcal{T}(P)$ have the same degree, namely rank $\hat{P}$. Thus, it suffices to show that for any $v \in \mathcal{T}(P)$ there exists $v_0 \in \mathcal{T}(P)$ with $v_0(-\infty) = \text{rank } \hat{P}$ such that $v - v_0 \in \mathcal{S}(P)$, that is, $v \geq v_0$ in $\mathcal{T}(P)$.

Let $v \in \mathcal{T}(P)$ and define $v_0 : \hat{P} \to \mathbb{N}$ by

$$v_0(x) = \max\{\text{depth } x, \text{rank } \hat{P} - v(-\infty) + v(x)\}$$

for all $x \in P$. Clearly, $v_0(-\infty) = \text{rank } \hat{P}$. We have to show that for any $y \gtrdot x$ in $\hat{P}$, we have $v_0(y) < v_0(x)$ and $v(y) - v_0(y) \leq v(x) - v_0(x)$. We first observe that our hypothesis implies that depth $x = \text{depth } y + 1$.

If $v_0(y) = \text{rank } \hat{P} - v(-\infty) + v(y)$, then $v_0(y) < \text{rank } \hat{P} - v(-\infty) + v(x) \leq v_0(y)$. If $v_0(y) = \text{depth } y$, then $v_0(y) < \text{depth } x \leq v_0(x)$.

For the second inequality, let us first take

$$v_0(x) = \text{depth } x > \text{rank } \hat{P} - v(-\infty) + v(x).$$

We get

$$\text{rank } \hat{P} - v(-\infty) + v(y) < \text{rank } \hat{P} - v(-\infty) + v(x) \leq \text{depth } x - 1 = \text{depth } y,$$

which implies that $v_0(y) = \text{depth } y$. Therefore, the inequality $v(y) - v_0(y) \leq v(x) - v_0(x)$ is equivalent to

$$v(y) - v(x) \leq \text{depth } y - \text{depth } x = -1,$$

which is obviously true. Now, let

$$v_0(x) = \text{rank } \hat{P} - v(-\infty) + v(x).$$

It follows that

$$v_0(y) - v_0(x) \geq (\operatorname{rank} \hat{P} - v(-\infty) + v(y)) - (\operatorname{rank} \hat{P} - v(-\infty) + v(x)) = v(y) - v(x)$$

which leads to the desired inequality. $\qquad\square$

Remark 2.6. By duality, one gets another sufficient condition for the levelness of the Hibi ring $R[L]$: If the subposet $\{y \in P : y \leq x\}$ of P is pure for all $x \in P$, then $R[L]$ is level.

In Figure 16 is displayed a poset P which shows that neither the condition given in Theorem 2.5 nor its dual are necessary for levelness. One may easily show that $R[\mathcal{I}(P)]$ is level either directly, by computing the minimal elements of $\mathcal{T}(P)$, or by using a computer to find the resolution of $R[\mathcal{I}(P)]$. However, the poset does not satisfy any of the sufficient conditions of being level.
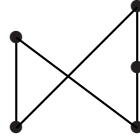

FIGURE 16. Butterfly poset

Remark 2.7. In [9], the poset of Figure 16 is called a *Butterfly poset.* One may consult [9] for more properties of butterfly posets.

A necessary condition for levelness was given in [9].

Theorem 2.8. [9, Theorem 4.1] *Suppose L is level. Then*

$$(4) \qquad\qquad \operatorname{height}(x) + \operatorname{depth}(y) \leq \operatorname{rank} \hat{P} + 1$$

for all $x, y \in P$ with $x \gtrdot y$.

Proof. Let $x, y \in P$ such that $x \gtrdot y$ and suppose that $\operatorname{height}(x) + \operatorname{depth}(y) > \operatorname{rank} \hat{P} + 1$. We have to show that L is not level.

By our assumption we get

$$\operatorname{height}(x) + \operatorname{depth}(y) > \operatorname{rank} \hat{P} + 1 \geq \operatorname{height}(x) + \operatorname{depth}(x) + 1,$$

and hence

$$\operatorname{depth}(y) > \operatorname{depth}(x) + 1.$$

We show that there exists a minimal element $w \in \mathcal{T}(P)$ with $w(-\infty) > \operatorname{rank} \hat{P}$. This then proves that L is not level.

Let $\operatorname{depth}(y) - \operatorname{depth}(x) - 1 = \alpha$. Then $\alpha > 0$. We define $v \colon \hat{P} \to \mathbb{N}$ as follows:

$$v(z) = \begin{cases} \operatorname{depth}(z) + \alpha, & \text{if } x \geq z,\ z \neq y, \\ \operatorname{depth}(z), & \text{otherwise.} \end{cases}$$

Then $v \in \mathcal{T}(P)$. If $v \in \mathcal{T}(P)$ is minimal, then we are done, since

$$v(-\infty) = \operatorname{depth}(-\infty) + \alpha = \operatorname{rank} \hat{P} + \alpha \geq \operatorname{rank} \hat{P} + 1.$$

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The last inequality follows from the fact that $\alpha > 0$.

On the other hand, if v is not minimal in $\mathcal{T}(P)$, then there exists a minimal element $w \in \mathcal{T}(P)$ with $v - w \in \mathcal{S}(P)$. It follows that

$$0 \leq v(x) - w(x) \leq v(y) - w(y) = \text{depth}(y) - w(y) \leq 0.$$

Hence

$$w(x) = v(x) = \text{depth}(x) + \alpha = \text{depth}(x) + \text{depth}(y) - \text{depth}(x) - 1 = \text{depth}(y) - 1.$$

Let

$$x = z_0 > z_1 > \cdots > z_k = -\infty$$

be a chain whose length is $\text{height}(x)$. Then

$$w(x) < w(z_1) < \cdots < w(z_k) = w(-\infty),$$

which implies that

$$w(-\infty) \geq w(x) + \text{height}(x) = (\text{depth}(y) - 1) + \text{height}(x) > \text{rank} \, \hat{P}.$$

$\square$

In the next subsection, we will see that, for a class of planar lattices, condition (4) is also sufficient for the level property of the Hibi ring.

2.4. **Regular hyper–planar lattices.** Hyper-planar lattices generalize the planar lattices. They were introduced in [9].

Definition 2.9. Let L be a finite distributive lattice and P its poset of join-irreducible elements. The lattice L is called a *hyper-planar lattice*, if P as a set is the disjoint union of chains $C_1, \ldots, C_d$, where each C_i is a maximal chain in P. We call such a chain decomposition *canonical*.

For $d = 2$, we recover simple planar lattices.

A canonical chain decomposition of the poset P of join-irreducible elements for a hyper-planar lattice L is, in general, not uniquely determined. However, if $C_1 \cup C_2 \cup \cdots \cup C_s$ and $D_1 \cup D_2 \cup \cdots \cup D_t$ are canonical chain decompositions of P, then $s = t$. Indeed, let $\max(Q)$ denote the set of maximal elements of a finite poset Q. Then

$$\tag{5} \begin{aligned} \max(P) &= \max(C_1) \cup \max(C_2) \cup \cdots \cup \max(C_s) \\ &= \max(D_1) \cup \max(D_2) \cup \cdots \cup \max(D_t). \end{aligned}$$

Let $\max(C_i) = \{x_i\}$ for $i = 1, \ldots, s$ and $\max(D_i) = \{y_i\}$ for $i = 1, \ldots, t$. Then the elements x_i as well as the elements y_i are pairwise distinct, and it follows from (5) that

$$\{x_1, x_2, \ldots, x_s\} = \{y_1, y_2, \ldots, y_t\},$$

Hence, t follows that $s = t$.

One would even expect the equality

$$\tag{6} \{\ell(C_1), \ell(C_2), \ldots, \ell(C_s)\} = \{\ell(D_1), \ell(D_2), \ldots, \ell(D_t)\},$$

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as multisets. However, this is not the case. The poset P displayed in Figure 17 has the following two canonical chain decompositions

$$C_1 = a < b < c < d < e < f, \quad C_2 = g < h < i < j < k < l,$$

and

$$D_1 = a < b < i < e < f, \quad D_2 = g < h < c < d < j < k < l.$$

We have $\ell(C_1) = \ell(C_2) = 5$, while $\ell(D_1) = 4$ and $\ell(D_2) = 6$.
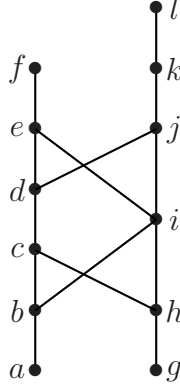

FIGURE 17

In order to guarantee that equality (6) is satisfied we have to add an extra condition on the hyper-planar lattice.

Definition 2.10. The lattice $L = \mathcal{I}(P)$ is called *regular hyper-planar*, if, for any canonical chain decomposition $C_1 \cup C_2 \cup \ldots \cup C_d$ of P, and for all $x < y$ with $x \in C_i$ and $y \in C_j$ it follows that $\text{height}_{C_i}(x) < \text{height}_{C_j}(y)$.

In the next corollary we give some properties of regular hyper-planar lattices. First we need the following result.

Lemma 2.11. [9, Lemma 3.1] *Let L be a regular hyper-planar lattice and $C_1 \cup \ldots \cup C_d$ a canonical chain decomposition of P. Then, for all i and $x \in C_i$, we have $\text{height}_{C_i}(x) = \text{height}_P(x)$.*

Proof. We apply induction on $\text{height}_P(x)$. If $\text{height}_P(x) = 0$, then there is nothing to show. Assume that $\text{height}_P(x) > 0$ and let $y \in P$ with $x \gtrdot y$ with $\text{height}_P(y) = \text{height}_P(x) - 1$. Let us assume that $y \in C_j$. Since $\text{height}_P(y) = \text{height}_P(x) - 1$, by the inductive hypothesis we obtain

$$\text{height}_P(x) - 1 = \text{height}_P(y) = \text{height}_{C_j}(y) < \text{height}_{C_i}(x) \leq \text{height}_P(x).$$

This yields the desired conclusion. $\qquad\qquad\square$

Corollary 2.12. [9, Corollary 3.2] *Let L be a regular hyper-planar lattice with the distinct canonical chain decompositions $C_1 \cup \ldots \cup C_d$ and $D_1 \cup D_2 \cup \cdots \cup D_d$ of P. Then*

    (a) $\{\ell(C_1), \ell(C_2), \ldots, \ell(C_d)\} = \{\ell(D_1), \ell(D_2), \ldots, \ell(D_d)\}$, *as multisets.*
    (b) $\text{rank } P = \max\{\ell(C_1), \ldots, \ell(C_d)\}$.

(c) $\mathrm{height}(x) + \mathrm{depth}(x) = \mathrm{rank}\,\hat{P}$ *for all* $x \in C_i$ *with* $\ell(C_i) = \mathrm{rank}\,P$.

Proof. Let $\max(C_i) = \{x_i\}$ and $\max(D_i) = \{y_i\}$ for $i = 1, \dots, t$. We have alreday seen that

$$\{x_1, x_2, \dots, x_d\} = \{y_1, y_2, \dots, y_d\},$$

Therefore,

$$\{\mathrm{height}_P(x_1), \mathrm{height}_P(x_2), \dots, \mathrm{height}_P(x_d)\} = \{\mathrm{height}_P(y_1), \mathrm{height}_P(y_2), \dots, \mathrm{height}_P(y_d)\},$$

as multi-sets. By Lemma 2.11, $\mathrm{height}_P(x_i) = \ell(C_i)$ and $\mathrm{height}_P(y_i) = \ell(D_i)$. On the other hand, $\mathrm{rank}\,P = \max\{\mathrm{height}_P(x_1), \mathrm{height}_P(x_2), \dots, \mathrm{height}_P(x_d)\}$. Then we have proved (a) and (b).

In order to prove (c), we observe that

$$\begin{aligned}
\mathrm{rank}\,\hat{P} &= \ell(\hat{C}_i) = \mathrm{height}_{\hat{C}_i}(x) + \mathrm{depth}_{\hat{C}_i}(x) \\
&\leq \mathrm{height}(x) + \mathrm{depth}(x) \leq \mathrm{rank}\,\hat{P}.
\end{aligned}$$

$\square$

In the next theorem we present the characterization of the regular hyper-planar lattices which are pseudo-Gorenstein.

Theorem 2.13. [9, Theorem 3.3] *Let L be a regular hyper-planar lattice and $C_1 \cup \dots \cup C_d$ a canonical chain decomposition of P. Then L is pseudo-Gorenstein if and only if all C_i have the same length.*

Proof. Suppose all the chains C_i have the same length. Then Corollary 2.12 implies that $\ell(C_i) = \mathrm{rank}\,\hat{P}$ for all i. Let $x \in P$. Then $x \in C_i$ for some i, and hence $\mathrm{height}(x) + \mathrm{depth}(x) = \mathrm{rank}\,\hat{P}$, by Corollary 2.12. Therefore, by Theorem 2.4, L is pseudo-Gorenstein.

Conversely, suppose that not all C_i have the same length. Then Corollary 2.12 implies that there exists one C_i with $\ell(C_i) < \mathrm{rank}\,P$. As in the proof of Theorem 2.4 we consider the strictly order reversing function $v(x) = \mathrm{depth}(x)$ and $v'(x) = \mathrm{rank}\,\hat{P} - \mathrm{height}(x)$. Let $x = \max(C_i)$. Then $v(x) = 1$ and, since L is regular, $v'(x) = \mathrm{rank}\,\hat{P} - (\ell(C_i) + 1) > \mathrm{rank}\,\hat{P} - \mathrm{rank}\,P - 1 = 1$. This shows that L is not pseudo-Gorenstein. $\square$

Examples 2.14. 1. For the poset P from Figure 18, the lattice $L = \mathcal{I}(P)$ is pseudo-Gorenstein since P satisfies the condition of Theorem 2.13 and it is not Gorenstein since P is not pure.
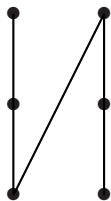
FIGURE 18

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2. The lattice $L = \mathcal{I}(P)$ where P is the regular planar poset displayed in Figure 16 is not pseudo-Gorenstein.

The next theorem shows that, for regular planar lattices, the necessary condition given in Theorem 2.8 is also sufficient for the levelness of the Hibi ring. Before stating this theorem we need a preparatory result.

Lemma 2.15. *Let L be a regular planar lattice. Let $C_1 \cup C_2$ be a canonical chain decomposition of P, and assume that $\ell(C_1) = \operatorname{rank} P$ (cf. Corollary 2.12). Suppose that P satisfies inequality (4) given in Theorem 2.8. Then, for every minimal element $v \in \mathcal{T}(P)$, we have $v(\max(C_1)) = 1$.*

Proof. Let $v \in \mathcal{T}(P)$ be a minimal element and assume that $v(\max(C_1)) > 1$. Then $v(z) \geq \operatorname{depth}(z) + 1$ for all $z \in C_1$.
Let

$$v'(x) = \begin{cases} v(x) - 1, & \text{if } v(x) \geq \operatorname{depth}(x) + 1 \quad \text{(I)}, \\ v(x), & \text{if } v(x) = \operatorname{depth}(x) \quad\quad \text{(II)}, \end{cases}$$

for all $x \in \hat{P}$.

We show that $v' \in \mathcal{T}(P)$ and $v - v' \in \mathcal{S}(P)$. Since $v' \neq v$, this will then show that v is not minimal, a contradiction. Indeed, to see that $v' \in \mathcal{T}(P)$ we have to show that $v'(x) < v'(y)$ for all $x \gtrdot y$. If both x and y satisfy (I) or (II), then the assertion is trivial. If x satisfies (I) and y satisfies (II), then $v'(x) = v(x) - 1 < v(y) = v'(y)$, and if x satisfies (II) and y satisfies (I), then $v(x) = \operatorname{depth}(x) \leq \operatorname{depth}(y) - 1 \leq v(y) - 2$. Hence $v(x) < v(y) - 1$, and this implies that $v'(x) < v'(y)$.

It remains to be shown that $v - v' \in \mathcal{S}(P)$ which amounts to prove that $v(x) - v'(x) \leq v(y) - v'(y)$ for all $x \gtrdot y$. For this we only need to show that we cannot have $v'(x) = v(x) - 1$ and $v(y) = v'(y)$, or, equivalently, that $v(x) \geq \operatorname{depth}(x) + 1$ and $v(y) = \operatorname{depth}(y)$ is impossible.

Assume to the contrary that there exist $x \gtrdot y$ with $v(x) \geq \operatorname{depth}(x) + 1$ and $v(y) = \operatorname{depth}(y)$. Then $y \notin C_1$ since $v(z) \geq \operatorname{depth}(z) + 1$ for all $z \in C_1$. Thus, we may either have $x \in C_1$ and $y \in C_2$, or $x, y \in C_2$.

In the first case, since $\operatorname{height}(x) + \operatorname{depth}(y) \leq \operatorname{rank} \hat{P} + 1$ by assumption, and since $\operatorname{rank} \hat{P} = \operatorname{height}(x) + \operatorname{depth}(x)$ due to the regularity of L (see Corollary 2.12), we get $\operatorname{depth}(y) \leq \operatorname{depth}(x) + 1 \leq v(x) < v(y)$, a contradiction.

Finally, let $x, y \in C_2$. Since $v(x) < v(y)$, it follows that $\operatorname{depth}(y) > \operatorname{depth}(x) + 1$. Therefore, the longest chain from y to ∞ cannot pass through x. This implies that there exists $z \in C_1$ with $z \gtrdot y$. As in the first case, we then deduce that $v(y) > \operatorname{depth}(y)$. So we get again a contradiction. $\square$

Theorem 2.16. *Let L be a regular planar lattice. Then the following conditions are equivalent:*

(a) *L is level;*
(b) *$\operatorname{height}(x) + \operatorname{depth}(y) \leq \operatorname{rank} \hat{P} + 1$ for all $x, y \in P$ with $x \gtrdot y$;*
(c) *for all $x, y \in P$ with $x \gtrdot y$, either $\operatorname{depth}(y) = \operatorname{depth}(x) + 1$ or $\operatorname{height}(x) = \operatorname{height}(y) + 1$.*

Proof. (a) $\Rightarrow$ (b) follows from Theorem 2.8.

(b) $\Rightarrow$ (c): Let $C_1 \cup C_2$ be a canonical chain decomposition of P with $|C_1| \geq |C_2|$. If $x, y \in C_1$ or $x, y \in C_2$, then, by Lemma 2.11, it follows that $\mathrm{height}(x) = \mathrm{height}(y) + 1$.

Next suppose that $x \in C_1$. Since L is regular, we may apply Corollary 2.12 and conclude that $\mathrm{height}(x) + \mathrm{depth}(x) = \mathrm{rank}\,\hat{P}$. Thus, by (b), we get $\mathrm{depth}(y) \leq \mathrm{depth}(x) + 1$. On the other hand, it is clear that $\mathrm{depth}(y) \geq \mathrm{depth}(x) + 1$. So that $\mathrm{depth}(y) = \mathrm{depth}(x) + 1$. Finally, if $y \in C_1$, then, by Corollary 2.12, we have $\mathrm{height}(y) + \mathrm{depth}(y) = \mathrm{rank}\,\hat{P}$. As in the previous case, we conclude that $\mathrm{height}(x) = \mathrm{height}(y) + 1$.

(c) $\Rightarrow$ (b): If $\mathrm{depth}(y) = \mathrm{depth}(x) + 1$, then $\mathrm{height}(x) + \mathrm{depth}(y) = \mathrm{height}(x) + \mathrm{depth}(x) + 1 \leq \mathrm{rank}\,\hat{P} + 1$, and if $\mathrm{height}(x) = \mathrm{height}(y) + 1$, then $\mathrm{height}(x) + \mathrm{depth}(y) = \mathrm{height}(y) + \mathrm{depth}(y) + 1 \leq \mathrm{rank}\,\hat{P} + 1$.

(b) $\Rightarrow$ (a): As in Lemma 2.15 we let $C_1 \cup C_2$ be a canonical chain decomposition of P, and may assume that $\ell(C_1) = \mathrm{rank}\,P \geq \ell(C_2)$. Let v be minimal in $\mathcal{T}(P)$. We will show that there exists $v' \in \mathcal{T}(P)$ with $v'(-\infty) = \mathrm{rank}\,\hat{P}$ and such that $v - v' \in \mathcal{S}(P)$. Since v is a minimal generator it follows that $v = v'$, thus $v(-\infty) = \mathrm{rank}\,\hat{P}$. Consequently, it follows that all the minimal generators of ω_L have the same degree.

In order to construct v' we consider the subposet Q of P which is obtained from P by removing the maximal elements $\max(C_1)$ and $\max(C_2)$. We define on $\hat{Q}$ the strictly order reversing function u by $u(\infty) = 0$, and $u(z) = v(z) - 1$ for all other $z \in \hat{Q}$. We notice that the ideal lattice of Q is again a regular planar lattice satisfying (b). Indeed, assume that there exist $x > y$ with $x, y \in Q$ such that $\mathrm{height}_{\hat{Q}}(x) + \mathrm{depth}_{\hat{Q}}(y) > \mathrm{rank}\,\hat{Q} + 1 = \mathrm{rank}\,\hat{P}$. Since $\mathrm{height}_{\hat{Q}}(x) = \mathrm{height}(x)$ and $\mathrm{depth}(y) = \mathrm{depth}_{\hat{Q}}(y) + 1$, it follows that

$$\mathrm{height}(x) + \mathrm{depth}(y) = \mathrm{height}_{\hat{Q}}(x) + \mathrm{depth}_{\hat{Q}}(y) + 1 > \mathrm{rank}\,\hat{P} + 1,$$

a contradiction.

Therefore, by induction on the rank we may assume that the ideal lattice of Q is level. Hence, there exists $w \in \mathcal{T}(Q)$ with $w(-\infty) = \mathrm{rank}\,\hat{Q} = \mathrm{rank}\,\hat{P} - 1$ and such that $u - w \in \mathcal{S}(Q)$. Set $v'(z) = 1 + w(z)$ for all $z \in A = Q \cup \{-\infty\}$. Then v' is a strictly order reversing function on A with $v'(-\infty) = \mathrm{rank}\,\hat{P}$ and such that $v - v'$ is order reversing on A. It remains to define $v'(C_i)$ for $i = 1, 2$ in a way such that $v' \in \mathcal{T}(P)$ and $v - v' \in \mathcal{S}(P)$. We have to set $v'(\max(C_1)) = 1$ since $v(\max(C_1)) = 1$, and of course $v'(\infty) = 0$. Let $x = \max(C_2)$ and let $z \in C_2$ be the unique element with $x \gtrdot z$. We set $v'(x) = v(x) - u(z) + w(z) = v(x) - v(z) + 1 + w(z)$, and claim that this v' has the desired properties. Indeed, $v'(x) = v(x) - (v(z) - 1 - w(z)) \leq v(x)$ and $v'(x) < 1 + w(z) = v'(z)$, since $v(x) < v(z)$. If z is the only element covered by x, we are done. Otherwise, there exists $y \in C_1$ with $x \gtrdot y$ and it remains to be shown that $v'(y) > v'(x) = v(x) - v(z) + 1 + w(z)$. Suppose we know that $\mathrm{depth}_{\hat{Q}}(y) \geq w(z)$, then

$$v'(y) = w(y) + 1 \geq \mathrm{depth}_{\hat{Q}}(y) + 1 > w(z) \geq v'(x),$$

as desired, since $v(x) - v(z) + 1 \leq 0$. Thus in order to complete the proof we have to show that $\operatorname{depth}_{\hat{Q}}(y) \geq w(z)$. Since the ideal lattice of $\hat{Q}$ is regular, this is equivalent to showing that

$$(7) \qquad\qquad w(z) \leq \operatorname{rank} \hat{Q} - \operatorname{height}_{\hat{Q}}(y).$$

The assumption (b) and Corollary 2.12(c) imply that

$$\operatorname{height}(x) + \operatorname{depth}(y) \leq \operatorname{rank} \hat{P} + 1 = \operatorname{height}(y) + \operatorname{depth}(y) + 1,$$

so that $\operatorname{height}(x) \leq \operatorname{height}(y) + 1$. This yields

$$(8) \qquad\qquad \operatorname{height}(x) = \operatorname{height}(y) + 1$$

since $\operatorname{height}(x) \geq \operatorname{height}(y) + 1$ always holds.

On the other hand, since L is regular, Lemma 2.11 implies that $\operatorname{height}_P(x) = \operatorname{height}_{C_2}(x) = \operatorname{height}_{C_2}(z) + 1 = \operatorname{height}_P(z) + 1$. This implies that $\operatorname{height}(x) = \operatorname{height}(z) + 1$. So together with (8) we then conclude that $\operatorname{height}(y) = \operatorname{height}(z)$. Since $\operatorname{height}_{\hat{Q}}(y) = \operatorname{height}(y)$ and $\operatorname{height}(z) = \operatorname{height}_{\hat{Q}}(z)$, inequality (7) becomes $w(z) \leq \operatorname{rank} \hat{Q} - \operatorname{height}_{\hat{Q}}(z)$, and since $w(-\infty) = \operatorname{rank} \hat{Q}$, this inequality indeed holds. This completes the proof of the theorem. $\qquad\square$

Remark 2.17. We do not know any example showing that condition (b) of the above theorem is not sufficient for the levelness of the Hibi ring. We conjecture that condition (b) in Theorem 2.16 is also sufficient for any distributive lattice.

At the end of this subsection we go back to Hibi's examples of Figure 15. They correspond to the two posets displayed in Figure 19. It is easily seen that the left side poset which corresponds to the level lattice in Figure 15 is not regular and of course satisfies condition (b) in the above theorem. The right side poset corresponds to the non-level lattice in Figure 15 and it does not satisfy condition (b) in Theorem 2.16.
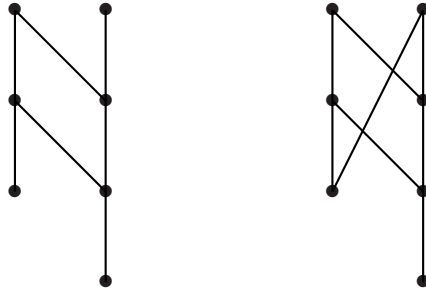

FIGURE 19

2.5. **Level and pseudo-Gorenstein generalized Hibi rings.** In Subsection 1.5 we have presented the construction of the generalized Hibi ring $R_r(P)$. Here $r \geq 2$ is an integer and P is a finite poset. We have seen in Theorem 1.35 that $R_r(P)$ is the classical Hibi ring of the lattice L_r whose poset of join-irreducible elements is $P_r = P \times Q_{r-1}$. This identification allowed us to prove that the generalized Hibi ring is Gorenstein if and only if P is pure.

In the next theorem, following [9, Section 5], we investigate some other properties of $R_r(P)$.

Theorem 2.18. [9, Theorem 5.1] *Let P be a finite poset and $r \geq 2$ an integer. Let $L = \mathcal{I}(P)$ and $L_r = \mathcal{I}(P_r)$. Then*

(a) $\operatorname{type} R[L] \leq \operatorname{type} R[L_r]$;
(b) *The ring $R[L]$ is pseudo-Gorenstein if and only if $R[L_r]$ is pseudo-Gorenstein.*
(c) *If $R[L_r]$ is level, then $R[L]$ is level.*

Proof. (a) We know that, for a distributive lattice L, type $R[L]$ is equal to the number of the minimal generators of ω_L, thus, $\operatorname{type} R[L] = |\min \mathcal{T}(P)|$ where $\min \mathcal{T}(P)$ denotes the set of minimal elements in $\mathcal{T}(P)$. Therefore, in order to prove (a), it suffices to find an injective map $\min \mathcal{T}(P) \to \min \mathcal{T}(P_r)$. We define $\varepsilon : \min \mathcal{T}(P) \to \min \mathcal{T}(P_r)$ as follows. If $v \in \min \mathcal{T}(P)$, then $\varepsilon(v)(x, i) = v(x) + (r - 1 - i)$ and $\varepsilon(v)(\infty) = 0, \varepsilon(v)(-\infty) = v(-\infty) + (r - 2)$. One easily checks that $v' = \varepsilon(v) \in \mathcal{T}(P_r)$. In order to show that $\varepsilon(v) \in \min \mathcal{T}(P_r)$, we prove that if $u \in \mathcal{T}(P_r)$ and $v' - u \in \mathcal{S}(P_r)$, then $v' = u$.

For any $w \in \mathcal{T}(P_r)$ and for $i \in [r - 1]$ we define the function w_i on $\hat{P}$ as follows:

$$w_i(x) = w(x, i) - (r - 1 - i) \quad \text{for all} \quad x \in P,$$

$w_i(\infty) = 0$ and $w_i(-\infty) = \max\{w_i(x) : x \in P\} + 1$. Then $w_i \in \mathcal{T}(P)$.

Since $v' - u \in \mathcal{S}(P_r)$ it follows that $v - u_i = v_i' - u_i \in \mathcal{S}(P)$. Since $v \in \min \mathcal{T}(P)$ we get $v = u_i$ for all i. This shows that $v' = u$.

It remain to show that ε is injective. Let $v, w \in \min \mathcal{T}(P)$ with $\varepsilon(v) = \varepsilon(w)$. By the definition of ε we get immediately $v = w$.

(b) By Theorem 2.4, $R[L_r]$ is pseudo-Gorenstein if and only if, for all $x \in P$,

$$\operatorname{height}_{\hat{P}_r} x + \operatorname{depth}_{\hat{P}_r} x = \operatorname{rank} \hat{P}_r.$$

We will show that, for all $x \in P$ and $1 \leq i \leq r - 1$,

$$(9) \quad \operatorname{height}_{\hat{P}_r} x = \operatorname{height}_{\hat{P}} x + i - 1 \text{ and } \operatorname{depth}_{\hat{P}_r} x = \operatorname{depth}_{\hat{P}} x + (r - i - 1).$$

In particular, we get $\operatorname{rank} \hat{P}_r = \operatorname{rank} \hat{P} + (r - 2)$. These equalities will then imply that $R[L_r]$ is pseudo-Gorenstein if and only if $\operatorname{height}_{\hat{P}} x + \operatorname{depth}_{\hat{P}_r} x = \operatorname{rank} \hat{P}$, that is, if and only if $R[L]$ is pseudo-Gorenstein.

We will prove only the first equality in (9). The other one may be proved in a similar way. If $\operatorname{height}_{\hat{P}} x = 1$, then we have nothing to prove since x is a minimal element in P and $i = 1$. Let $\operatorname{height}_{\hat{P}} x > 1$ and $x = x_0 > x_1 > \cdots > x_h > -\infty$ be a maximal chain in $\hat{P}$ of length equal to $\operatorname{height}_{\hat{P}} x$. Then

$$(x, i) = (x_0, i) > (x_1, i) > \cdots > (x_h, i) > (x_h, i - 1) > \cdots > (x_h, 1) > -\infty$$

is a chain of length $\operatorname{height}_{\hat{P}} x + i - 1$ in $\hat{P}_r$. Therefore, we have $\operatorname{height}_{\hat{P}_r} x \geq \operatorname{height}_{\hat{P}} x + i - 1$. For the other inequality, we proceed by induction on $\operatorname{height}_{\hat{P}} x$. Let $(x, i) = z_0 > z_1 > \cdots > z_t > -\infty$ be a chain of length $\operatorname{height}_{\hat{P}} x$ in $\hat{P}_r$. Then either $z_1 = (y, i)$ where $x > y$ in P or $z_1 = (z, i - 1)$. By the inductive hypothesis, in the first case $\operatorname{height}_{\hat{P}_r}(z_1) \leq \operatorname{height}_{\hat{P}} y + (i - 1) \leq \operatorname{height}_{\hat{P}} x + i - 2$, and, in

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the second case, $\text{height}_{\hat{P}_r}(z_1) \leq \text{height}_{\hat{P}} x + (i-2)$. In both cases it follows that $\text{height}_{\hat{P}_r} x \leq \text{height}_{\hat{P}} x + i - 1$.

(c) Suppose that L is not level. Then there exists $v \in \min \mathcal{T}(P)$ with $v(-\infty) > \text{rank}\, \hat{P}$. Then $\varepsilon(v)$, as defined in the proof of part (a), belongs to $\min \mathcal{T}(P_r)$ and

$$\varepsilon(v)(-\infty) = v(-\infty) + (r-2) > \text{rank}\, \hat{P} + (r-2) = \text{rank}\, \hat{P}_r.$$

This shows that L_r is not level. □

3. The regularity of Hibi rings

Let L be a distributive lattice and P its subset of join-irreducible elements. We assume that $|P| = n$. Hence, $\text{rank}\, L = n$. The ring $R[L]$ is a standard graded algebra with the presentation $R[L] = S/I_L$ where $S = K[\{x_\alpha : \alpha \in L\}]$ and

$$I_L = (x_\alpha x_\beta - x_{\alpha \cap \beta} x_{\alpha \cup \beta} : \alpha \in L, \alpha, \beta \text{ incomparable.})$$

Not so much is known about the S-resolution of the Hibi ring $R[L]$.

One may easily compute the projective dimension of $R[L]$. Since $R[L]$ is Cohen-Macaulay, $\text{proj dim}\, R[L] = |L| - \dim R[L]$. Since $R[L]$ and $S/\text{in}_<(I_L)$ have the same Hilbert series, it follows that $\dim R[L] = \dim(S/\text{in}_<(I_L))$. As we have already seen in Subsection 1.3.1, $S/\text{in}_<(I_L)$ is the Stanley-Resiner ring of the order complex of L. Since the facets of this complex have the cardinality equal to $|P| + 1$, we get $\dim(S/\text{in}_<(I_L)) = |P| + 1$. Therefore,

$$\text{proj dim}\, R[L] = |L| - |P| - 1.$$

Another important homological invariant of $R[L]$ is the regularity. In this section we present the formula for $\text{reg}\, R[L]$ following [10]. This can be given in terms of the poset P. In the second part of this section, we study Hibi rings with linear syszygies and with pure resolution for planar distributive lattices.

3.1. **The regularity of Hibi rings.** Before stating the formula for the regularity of $R[L]$, we explain how one may compute the regularity of a Cohen-Macaulay standard graded K–algebra. Let R be a Cohen-Macaulay standard graded K–algebra, say $R = T/I$ where $T = K[x_1, \ldots, x_n]$ and $I \subset T$ a graded ideal. The Hilbert series of R has the form $H_R(t) = Q(t)/(1-t)^{\dim R}$ where $Q(t) = \sum_{i \geq 0} h_i t^i \in \mathbb{Z}[t]$ with $Q(1) \neq 0$. The vector of the coefficients of Q, $h = (h_0, h_1, \ldots)$, is called the h–vector of R. As R is Cohen-Macaulay, one may find an R–regular sequence $\theta = \theta_1, \ldots, \theta_{\dim R}$ of linear forms. The rings R and $R/\theta R$ have the same h–vector and the same regularity [24, Theorem 20.2]. Since $\dim R/\theta R = 0$, we have $\text{reg}\, R/\theta R = \deg h$ [6, Exercise 20.18]. Consequently,

$$(10) \qquad \qquad \text{reg}\, R = \deg h.$$

The a–invariant $a(R)$ of R is defined as the degree of the Hilbert series of R; see [3, Definition 4.4.4]. Hence, we have $a(R) = \deg h - \dim R$. On the other hand, $a(R) = -\min\{i : (\omega_R)_i \neq 0\}$ where ω_R is the canonical module of R [3, Chapter 4]. Therefore,

$$(11) \qquad \qquad \text{reg}\, R = \dim R - \min\{i : (\omega_R)_i \neq 0\}.$$

Theorem 3.1. [10, Theorem 1.1] *Let $L = \mathcal{I}(P)$ be a distributive lattice and $R[L]$ its Hibi ring. Then* $\operatorname{reg} R[L] = |P| - \operatorname{rank} P - 1$.

Proof. We know that $\dim R[L] = |P| + 1$. By equality (11), we need to compute the initial degree of ω_L. In other words, we have to compute $\min\{v(-\infty) : v \in \mathcal{T}(P)\}$. We have seen in Subsection 1.4 that $v(-\infty) \geq \operatorname{rank} \hat{P} = \operatorname{rank} P + 2$. On the other hand, $\operatorname{depth} \in \mathcal{T}(P)$ and $\operatorname{depth}(-\infty) = \operatorname{rank} \hat{P}$. Therefore, $\min\{v(-\infty) : v \in \mathcal{T}(P)\} = \operatorname{rank} P + 2$. This implies that $\operatorname{reg} R[L] = |P| - \operatorname{rank} P - 1$. $\qquad\square$

A combinatorial proof of the above theorem can be found in [10].

As a direct consequence of Theorem 3.1, we may characterize the lattices L for which $R[L]$ has a linear resolution. This characterization was first obtained in [13]. We can restrict to simple lattices. Recall that $L = \mathcal{I}(P)$ is called *simple* if there is no $p \in P$ with the property that any element of P is comparable to p. In lattice, this means that there are no elements $\alpha < \beta$ in L such that any element $\gamma \in L$ satisfies either $\gamma \geq \beta$ or $\gamma \leq \alpha$. In the sequel, we may assume without any restrictions that L is simple. Indeed, if L is not simple, we let P' to be the subposet of P which is obtained by removing a vertex $p \in P$ which is comparable with any other vertex of P and set $L' = \mathcal{I}(P')$. Then I_L and $I_{L'}$ have the same regularity. Indeed, $|P'| = |P| - 1$, and since any maximal chain of P passes through p, it also follows that $\operatorname{rank} P' = \operatorname{rank} P - 1$. Thus the assertion follows from Theorem 3.1.

Corollary 3.2. *Let $L = \mathcal{I}(P)$ be a finite simple distributive lattice. Then $R[L]$ has a linear resolution if and only if P is the sum of a chain and an isolated element.*

Proof. The Hibi ring $R[L]$ has an S–linear resolution if and only if $\operatorname{reg} R[L] = 1$. By Theorem 3.1, this is equivalent to $|P| - \operatorname{rank} P = 2$. Hence, apart of a chain, P contains just one element. This element cannot be comparable to any element of the chain since the lattice is simple. $\qquad\square$

Theorem 3.1 allows the characterization of several other Hibi rings.

Extremal Cohen-Macaulay and Gorenstein algebras appeared in [26] and [27]. In [22], nearly extremal Cohen-Macaulay and Gorenstein algebras were defined. Let $R = T/I$ be a standard graded algebra. Here T is a polynomial ring over K n finitely many variables and $I \subset T$ is a graded ideal. Let $h = (h_0, \ldots, h_s)$ be the h–vector of R and assume that the initial degree of I is equal to p.

(1). Suppose that R is Cohen-Macaulay. Then $s \geq p - 1$. If $s = p - 1$ ($s = p$), then R is called (*nearly*) *extremal Cohen-Macaulay*.

(2). Suppose that R is Gorenstein. Then $s \geq 2(p - 1)$. If $s = 2(p - 1)$ (s=2p-1), then R is called (*nearly*) *extremal Gorenstein*.

Since $\operatorname{reg} R[L] = \deg h$, we may use Theorem 3.1 to characterize the simple lattices L (or, equivalently, the poset P) for which $R[L]$ is a (nearly) extremal Cohen-Macaulay or Gorenstein algebra. In our case, the initial degree of the presentation ideal of $R[L]$ is equal to 2. Therefore, we get:

(i). If $\operatorname{reg} R[L] = 1$ ($\operatorname{reg} R[L] = 2$), then $R[L]$ is (nearly) extremal Cohen-Macaulay. Thus, $R[L]$ is extremal Cohen-Macaulay if and only if $R[L]$ has a linear resolution.

In order to characterize the lattices L for which $\operatorname{reg} R[L] = 2$, we have to find all the posets P with $|P| = \operatorname{rank} P + 3$. This characterization was done in [10]. Let C be a maximal chain in P. Since $|P| = \operatorname{rank} P + 3$, it follows that there exist precisely two elements $q, q' \in P$ which do not belong to C. The only posets satisfying $|P| = \operatorname{rank} P + 3$ for which $L = \mathcal{I}(P)$ is simple are displayed in Figure 20.
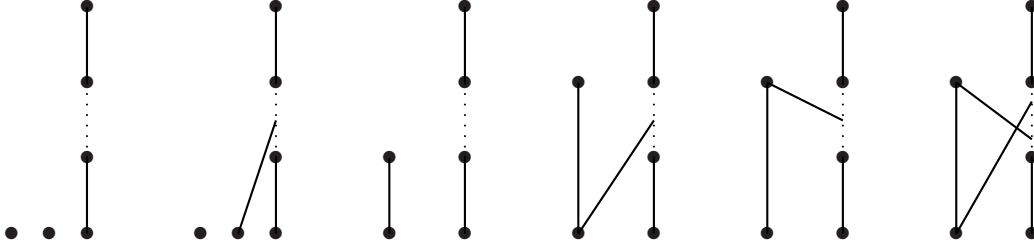

FIGURE 20

(ii). Let $R[L]$ be Gorenstein. By Theorem 1.29, P is a pure poset. The ring $R[L]$ is (nearly) extremal Goresnstein if $\operatorname{reg} R[L] = 2$ ($\operatorname{reg} R[L] = 3$). In the first case we get easily the poset P of one of the forms displayed in Figure 21; see also [10]. In the second case one obtains 12 (???) poset P for which $R[L]$ is nearly extremal Gorenstein.

FIGURE 21. Extremal Gorenstein

We end this subsection by a few comments on the regularity of Hibi rings for planar distributive lattices. We consider the infinite distributive lattice $\mathbb{N}^2$ with the partial order defined as $(i, j) \le (k, \ell)$ if $i \le k$ and $j \le \ell$. A *planar distributive lattice* is a finite sublattice L of $\mathbb{N}^2$ with $(0, 0) \in L$ which has the following property: for any $(i, j), (k, \ell) \in L$ there exists a chain $\mathfrak{c}$ in L of the form $\mathfrak{c} : x_0 < x_1 < \cdots < x_t$ with $x_s = (i_s, j_s)$ for $0 \le s \le t$, $(i_0, j_0) = (i, j)$, and $(i_t, j_t) = (k, \ell)$, such that $i_{s+1} + j_{s+1} = i_s + j_s + 1$ for all s. Planar distributive lattice are also called *two-sided ladders*.

In the planar case, we may compute the regularity of $R(L)$ in terms of the cyclic sublattices of L. A sublattice of L is called *cyclic* if it looks like in Figure 22 with some possible cut edges in between the squares. By a *square* in L we mean a sublattice with elements a, b, c, d such that $d > b > a$, $d > c > a$, and b, c are incomparable.

It is easily seen that for a cyclic lattice C with r squares we have $\operatorname{reg} R[C] = r$. Of course, this may be derived with the formula of Theorem 3.1, but we may give
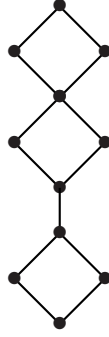
FIGURE 22. Cyclic sublattice

also a simpler argument as in [13]. I_C is generated by a regular sequence of length r since $\mathrm{in}_<(I_C)$ is generated by a regular sequence of monomials. Therefore, the Koszul complex of the generators of I_C is the minimal free resolution of $R[C]$ and, hence, $\mathrm{reg}\,R[C] = r$.

Theorem 3.3. [13] *Let L be a planar distributive lattice. Then $\mathrm{reg}\,R[L]$ equals the maximal number of squares in a cyclic sublattice of L.*

The interested reader may find the complete proof in [13]. Here we only mention that the proof uses combinatorial interpretations of the components of the h-vector of $R[L]$ given in [2, Section 2]. It turns out that $\deg h$ is equal to the maximal numbers of squares in a cyclic sublattice of L which explains the statement of the theorem.

The above theorem allows us, in relatively small examples, to *read* the regularity of $R[L]$ by looking at the Hasse diagram of L as in Figure 23.
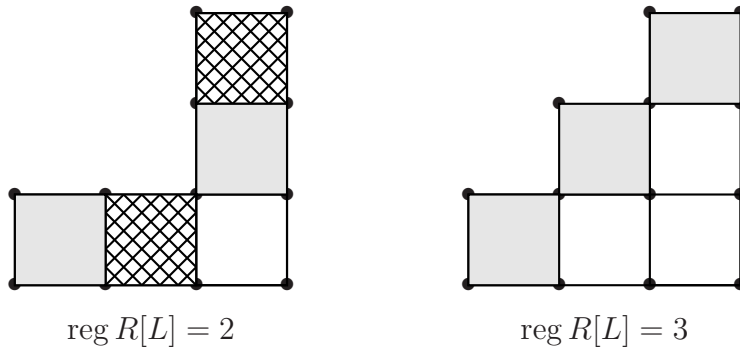

$$\mathrm{reg}\,R[L] = 2 \qquad\qquad \mathrm{reg}\,R[L] = 3$$

FIGURE 23

One could ask whether we can *read* as well the pseudo-Gorenstein property of $R[L]$ from the drawing of L. A rigorous answer to this question wsa given in [9]. Here, we briefly explain the picture of the pseudo-Gorensteiness without giving a formal proof. As we have seen in Proposition 2.3, $R[L]$ is pseudo-Gorenstein if and only if the leading coefficient of the numerator of the Hilbert series of $R[L]$ is equal to 1. According to the proof of [13, Theorem 4], this coefficient is equal to the number of cyclic sublattices of L with the largest number of squares. Hence, L (or $R[L]$)

is pseudo-Gorenstein if and only if it contains exactly one cyclic sublattice with maximum number of squares. For example, the lattice of the left side in Figure 23 is not pseudo-Gorenstein since, as we may see in figure, there are at least two cyclic sublattices with two squares, while the lattice displayed in the right side of the same figure is pseudo-Gorenstein.

3.2. **Hibi ideals with linear relations.** In the remaining part of this section we will restrict to planar distributive lattices. Even with this restriction, the calculation of all the graded Betti numbers of the Hibi ideals seems to be very difficult. In this subsection we aim at describing the shape of those planar distributive lattices L with the property that I_L has linear relations. We say that I_L has *linear relations* or that it is *linearly related* if $\beta_{1j} = 0$ for all $j \geq 4$.

The following lemma offers a major reduction in our study; see also [8, Corollary 1.4].

Lemma 3.4. *Let $I \subset T$ be a graded ideal in the polynomial ring T over a field K with finitely many indeterminates. If I has a quadratic Gröbner basis, then $\beta_{1j}(I) = 0$ for $j > 4$.*

Proof. By hypothesis, there exists a monomial order $<$ on T such that $\mathrm{in}_<(I)$ is generated in degree 2. Therefore, it follows from [18, Corollary 4] that $\beta_{1j}(\mathrm{in}_<(I)) = 0$ for $j > 4$. Since $\beta_{1j}(I) \leq \beta_{1j}(\mathrm{in}_<(I))$ (see, for example, [16, Corollary 3.3.3]), the desired conclusion follows. $\square$

The planar lattices may be viewed as convex polyominoes as well. For more information on this notion we refer the reader to [8]. All convex polyominoes whose ideals have linear relations were characterized in [8]. In this work, we follow the approach from [8], but we adapt some of the proofs to Hibi ideals for planar lattices. The main tool in our study is the squarefree divisor complex which allows the calculation of the multi-graded Betti numbers of a toric ideal.

We briefly recall the construction of the squarefree divisor complex which was introduced in [4]. Let K be a field and $H \subset \mathbb{N}^n$ an affine semigroup minimally generated by $h_1, \ldots, h_m$ where $h_i = (h_i(1), \ldots, h_i(n)) \in \mathbb{N}^n$. Let $K[H] \subset T = K[t_1, \ldots, t_n]$ be the semigroup ring assciated with H. Then $K[H] = K[u_1, \ldots, u_m]$ where $u_i = \prod_{j=1}^n t_j^{h_i(j)}$. Let $\varphi : R = K[x_1, \ldots, x_n] \to T$ be the K–algebra homomorphism induced by $x_i \mapsto u_i$ for $1 \leq i \leq n$, and I_H the kernel of φ. The ideal I_H is called the *toric ideal* of $K[H]$ or, simply, of H. To each variable x_i we assign the multidegree h_i. In this way, $K[H]$ and its toric ideal are $\mathbb{Z}^n$–graded R–modules. Thus I_H and $K[H]$ have $\mathbb{Z}^n$–graded minimal free resolutions. When all the monomials u_i have the same degree, then $K[H]$ may be viewed as a standard graded K–algebra by setting $\deg u_i = 1$ for all h. In this case, the degree of $t_1^{h(1)} \cdots t_n^{h(n)} \in K[H]$ will be denoted $|h|$.

Let $h \in H$. The squarefree divisor complex Δ_h is defined as follows. Its facets are the sets $F = \{i_1, \ldots, i_k\} \subset [m]$ such that $u_{i_1} \cdots u_{i_k} | t_1^{h(1)} \cdots t_n^{h(n)}$ in $K[H]$. Let $\tilde{H}_i(\Gamma, K)$ be the ith reduced simplicial homology of a simplicial complex Γ.[3]

Proposition 3.5 ([4]). *With the notation and assumptions introduced one has* $\mathrm{Tor}_i(K[H], K)_h \cong \tilde{H}_{i-1}(\Delta_h, K)$. *In particular,*

$$\beta_{ih}(K[H]) = \dim_K \tilde{H}_{i-1}(\Delta_h, K).$$

Let H' be a subsemigroup of H generated by a subset of the set of generators of H, and let R' be the polynomial ring over K in the variables x_i with h_i generator of H'. Furthermore, let $\mathbb{F}'$ the $\mathbb{Z}^n$-graded free R'-resolution of $K[H']$. Then, since R is a flat R'-module, $\mathbb{F}' \otimes_{S'} S$ is a $\mathbb{Z}^n$-graded free S-resolution of $S/I'_H S$. The inclusion $K[H'] \to K[H]$ induces a $\mathbb{Z}^n$-graded complex homomorphism $\mathbb{F}' \otimes_{S'} S \to \mathbb{F}$. Tensoring this complex homomorphism with $K = R/\mathfrak{m}$, where $\mathfrak{m}$ is the graded maximal ideal of R, we obtain the following sequence of isomorphisms and natural maps of $\mathbb{Z}^n$-graded K-modules

$$\mathrm{Tor}_i^{R'}(K[H'], K) \cong H_i(\mathbb{F}' \otimes_{R'} K) \cong H_i(\mathbb{F}' \otimes_{R'} R) \otimes_R K) \to H_i(\mathbb{F} \otimes_R K) \cong \mathrm{Tor}_i^R(K[H], K).$$

With an additional assumption on H' we get even more.

Corollary 3.6. [8, Corollary 2.3] *With the notation and assumptions introduced, let H' be a subsemigroup of H generated by a subset of the set of generators of H, and let h be an element of H' with the property that $h_i \in H'$ whenever $h - h_i \in H$. Then the natural K-vector space homomorphism* $\mathrm{Tor}_i^{R'}(K[H'], K)_h \to \mathrm{Tor}_i^R(K[H], K)_h$ *is an isomorphism for all i.*

For the proof of this corollary we refer to [8].

Definition 3.7. *Let $H \subset \mathbb{N}^n$ be an affine semigroup generated by $h_1, \ldots, h_m$. An affine subsemigroup $H' \subset H$ generated by a subset of $\{h_1, \ldots, h_m\}$ is called a homological pure subsemigroup of H if for all $h \in H'$ and all h_i with $h - h_i \in H$ it follows that $h_i \in H'$.*

In other words, H' is a homological pure subsemigroup of H if it satisfies the hypothesis of Corollary 3.6. We also say that $K[H']$ is a *homological pure subring* of $K[H]$. Corollary 3.6 has the following consequence.

Corollary 3.8. [8, Corollary 2.4] *Let H' be a homologically pure subsemigroup of H. Then*

$$\mathrm{Tor}_i^{R'}(K[H'], K) \to \mathrm{Tor}_i^R(K[H], K)$$

is injective for all i. In other words, if $\mathbb{F}'$ is the minimal $\mathbb{Z}^n$-graded free R'-resolution of $K[H']$ and $\mathbb{F}$ is the minimal $\mathbb{Z}^n$-graded free R-resolution of $K[H]$, then the complex homomorphism $\mathbb{F}' \otimes R \to \mathbb{F}$ induces an injective map $\mathbb{F}' \otimes K \to \mathbb{F} \otimes K$. In particular, any minimal set of generators of $\mathrm{Syz}_i(K[H'])$ is part of a minimal set of generators of $\mathrm{Syz}_i(K[H])$. Moreover, $\beta_{ij}(I_{H'}) \leq \beta_{ij}(I_H)$ for all i and j.

---

[3] For more information on the theory of simplicial complexes and their simplicial homology we refer the reader to [30] and [3, Chapter 5].

For the proof, see [8].

Let L be a planar distributive lattice. We may assume that $[(0,0),(m,n)]$ where m,n are some positive integers, is the smallest interval which contains L. In particular, we may assume that L contains the vertices of the squares $[(0,0),(1,1)]$ and $[(m-1,n-1),(m,n)]$. There is no loss of generality in this latter assumption since it simply means that that the poset P of the join-irreducible elements of L has two minimal and two maximal elements. If P has a unique minimal element, say p, then $R[\mathcal{I}(P)]$ and $R[\mathcal{I}(P \setminus \{p\})]$ have the same Betti numbers. The same happens when P contains a unique maximal element. We also may assume that $m,n \geq 2$. If, for instance, $n = 1$, then we know, by Theorem 3.2, that I_L has a linear resolution, thus, in particular, it has linear relations.

Let $A = \{i_1, \ldots, i_s\} \subset [m]$ be a set of integers and L_A be the subset of L obtained by removing all the elements (i,j) of L with $i \in A$. Then L_A is a sublattice of L. Indeed, if $(i,j),(k,\ell) \in L_A$, then $i,k \notin A$, thus $\min\{i,k\}$ and $\max\{i,k\}$ do not belong to A as well. Thus L_A is a sublattice of L. Analogously, we may consider the same procedure by using a subset $B \subset [n]$ and get a sublattice L_B of L. We call a sublattice of L obtained in one of the above ways an *induced sublattice* of L. Moreover, one easily sees that if L' is an induced sublattice of L, then $R[L']$ is a homological pure subring of $R[L]$. On the other hand, let us note that not any sublattice of L is an induced one.

Corollary 3.8 has the following consequence.

Corollary 3.9. *Let $A \subset [m]$ and $B \subset [n]$. With the above notation, we have*
$$\beta_{ij}(I_{L_A}) \leq \beta_{ij}(L) \text{ and } \beta_{ij}(I_{L_B}) \leq \beta_{ij}(L)$$
for all i,j. Moreover, each minimal relation of I_{L_A} or I_{L_B} is a minimal relation of I_L.

This corollary will be useful to isolate the Hibi ideals of planar lattices which have linear relations.

We begin with the following lemma which shows, in particular, that in order to get linear relations for I_L it is enough to consider L a simple lattice.

Lemma 3.10. *Let $L = \mathcal{I}(P) \subset [(0,0),(m,n)]$ be a planar distributive lattice which contains the vertices of the squares $[(0,0),(1,1)]$ and $[(m-1,n-1),(m,n)]$. If L is not simple, then $\beta_{14}(I_L) \neq 0$.*

Proof. The claim of the lemma is equivalent to $\beta_{24}(R[L]) \neq 0$. Since L is not simple, there exists $p \in P$ such that any other element of P is comparable to p. Let $P_1 = \{q \in P : q < p\}$ and $P_2 = \{q \in P : q > p\}$. Then P is the ordinal sum $P = P_1 \oplus \{p\} \oplus P_2$ and $R[\mathcal{I}(P)] \cong R[\mathcal{I}(P_1)] \otimes R[\mathcal{I}(P_2)]$. Let $\mathbb{F}_1 \to R[\mathcal{I}(P_1)] \to 0$ and $\mathbb{F}_2 \to R[\mathcal{I}(P_2)] \to 0$ be the minimal free resolutions of $R[\mathcal{I}(P_1)]$ and $R[\mathcal{I}(P_2)]$. Then the total complex of $\mathbb{F}_1 \otimes \mathbb{F}_2$ is the minimal S–free resolution of $R[L]$. This implies that $\beta_{24}(R[L]) \neq 0$ since $\beta_{12}(R[\mathcal{I}(P_1)]) \neq 0$ and $\beta_{12}(R[\mathcal{I}(P_2)]) \neq 0$. $\square$

The above lemma combined with Corollary 3.8 lead to the following type of arguments. Assume that, given a simple planar distributive lattice L, we may find a subset $A \subset [m]$ such that L_A is not simple and contains the extremal corners

$[(0,0),(1,1)]$ and $[(m-1,n-1),(m,n)]$. Then, it follows that I_{L_A} is not linearly related. This will imply that I_L is not linearly related, too.

The following theorem characterizes the simple planar distributive lattices L with linearly related Hibi ideals for $m, n \geq 3$. The case $m = 2$ or $n = 2$ is settled by the following lemma.

Lemma 3.11. *Let L be a simple planar distributive lattice $L \subset [(0,0),(m,n)]$ and assume that $m = 2$ or $n = 2$. If I_L is linearly related, then at most one of the vertices $(m,0)$ and $(0,n)$ do not belong to L.*

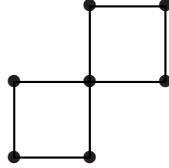*Proof.* Let us take, fir example, $n = 2$. If both vertices $(m,0)$ and $(0,2)$ do not

FIGURE 24

belong to L, then we find an induced sublattice of the forms displayed in Figure 24 which has the associated ideal not linearly related. $\qquad\square$

Theorem 3.12. *Let L be a simple planar distributive lattice, $L \subset [(0,0),(m,n)]$ with $m, n \geq 2$. The ideal I_L is linearly related if and only if the following conditions hold:*

(i) *At most one of the vertices $(m,0)$ and $(0,n)$ do not belong to L.*
(ii) *The vertices $(1,n-1)$ and $(m-1,1)$ belong to L.*

The only if part of the proof of this theorem follows from the following lemmas.

Lemma 3.13. *Let L be a lattice as in the statement of Theorem 3.12 and assume that both vertices $(m,0)$ and $(0,n)$ do not belong to L. Then $\beta_{14}(I_L) \neq 0$.*

Proof. Let $A = \{2,\ldots,m-2\}$ and L_A the corresponding induced lattice. The lattice L_A may be now framed in the interval $[(0,0),(3,n)]$. We choose now the set $B = \{2,\ldots,n-2\} \subset [n]$ and consider the induced sublattice L_{AB} of L_A. The lattice L_{AB} is isomorphic to one of the forms displayed in Figure 25:

In the first two cases, it is clear, by Lemma 3.10, that $\beta_{14}(I_{L_{AB}}) \neq 0$. In the last case, one may easily see that L_{AB} contains an induced cyclic sublattice with two squares, thus $\beta_{14}(I_{L_{AB}}) \neq 0$. Hence, by applying Corollary 3.9, we get $\beta_{14}(I_L) \neq 0$. $\qquad\square$

Lemma 3.14. *Let L be a lattice as in the statement of Theorem 3.12 and assume that $(m,0) \in L$ and $(0,n) \notin L$. If I_L is linearly related, then $(1,n-1) \in L$.*

Proof. Assume that $(1,n-1) \notin L$. We show that $\beta_{14}(I_L) \neq 0$. Proceeding as in the proof of Lemma 3.13, we get an induced sublattice L_{AB} of L which is displayed in Figure 26.

One checks with a computer that $\beta_{14}(I_{L_{AB}}) \neq 0$ which implies the desired conclusion. $\qquad\square$
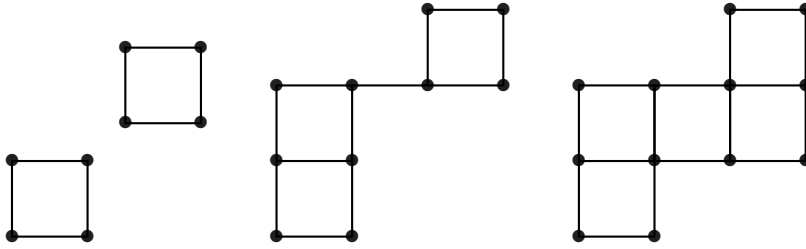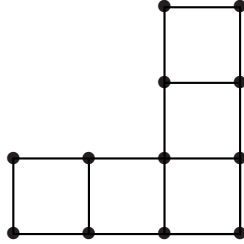
36

FIGURE 25

FIGURE 26

Proof of Theorem 3.12. Lemma 3.11, Lemma 3.13, and Lemma 3.14 complete the "only if" part of the proof.

It remains to prove that if L satisfies conditions (i) and (ii) in the statement of the theorem, then I_L is linearly related.

To begin with, we recall from [25] that the ring S/I_L which is isomorphic to $K[H]$ may be viewed as a semigroup ring $K[s_1, \ldots, s_m, t_1, \ldots, t_n]$ generated by the monomials $u_{ij} = s_i t_j$ where $(i, j) \in L$. With this interpretation of $K[H]$ in mind, we ill use Corollary 3.6 to show that I_L is linearly related.

Let $u = u_{i_1 j_1} u_{i_2 j_2} u_{i_3 j_3} u_{i_4 j_4}$ be an element of $K[H]$ viewed as a subring of the polynomial ring $K[s_1, \ldots, s_m, t_1, \ldots, t_n]$ and let $i = \min_q\{i_q\}, k = \max_q\{i_q\}, j = \min_q\{j_q\}$, and $\ell = \max_q\{j_q\}$. Therefore, all the points h_q lie in the (possible degenerate) rectangle $\mathcal{Q}$ of vertices $(i, j), (k, j), (i, \ell), (k, \ell)$. If $\mathcal{Q}$ is degenerate, that is, all the vertices of Q are contained in a vertical or horizontal line segment in L, then $\beta_{1h}(I_L) = 0$ since in this case the simplicial complex Δ_h is just a simplex. Let us now consider $\mathcal{Q}$ non-degenerate. If all the vertices of $\mathcal{Q}$ belong to L, then the interval $L' = [(i, j), (k, \ell)]$ is an induced sublattice of L. Therefore, by Corollary 3.6, we have $\beta_{1h}(I_L) = \beta_{1h}(I_{L'}) = 0$, the latter equality being true since L' is linearly related. The only case left to be discussed is that one when one corner of the rectangle $\mathcal{Q}$ does not belong to L. In this case one, one esaily sees that, by Corollary 3.6, $\beta_{14}(I_L)$ coincides with $\beta_{14}(I_{L'})$ where L' is an induced sublattice of L isomorphic to one displayed in Figure 27.

One may check with a computer algebra system that all lattices displayed in Figure 27 are linearly related, hence they do not have any relation in degree h. Just one final word for $m = 2$. In this case, we find an indiced sublattice of L isomorphic
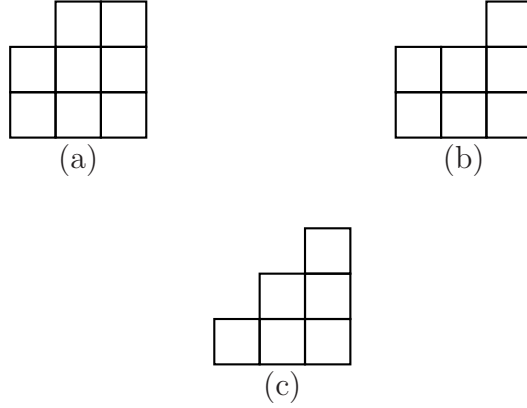
37

(a)   (b)

(c)

FIGURE 27

to an induced sublattice of L', hence, again, we do not find any relation of I_L in degree 4.

3.3. **Hibi ideals with pure resolutions.** Let L be a planar distributive lattice, $L \subset [(0,0),(m,n)]$ with $m, n \geq 1$.. As in the previous subsection, we assume that the squares $[(0,0),(1,1)]$ and $[(m-1,n-1)]$ belong to L. In the last part of this section we would like find under which conditions on L the ideal I_L has a pure resolution.

By Corollary 3.2, we know that I_L has a linear resolution if and only if $m = 1$ or $n = 1$. Therefore, we may consider $m, n \geq 2$.

We have already seen in Subsection 3.1 that if C is a cyclic lattice, then I_C has a pure resolution given by the Koszul complex of the sequence of its binomial generators. In addition, let us observe that if L is not simple, then $R[L]$ may be expressed as $R[L] \cong R[L_1] \otimes R[L_2]$ where $L_1 = \mathcal{I}(P_1)$ and $L_2 = \mathcal{I}(P_2)$ with P_1, P_2 as they have been defined in the proof of Lemma 3.10. Hence, if at least one of the ideals I_{L_1} or I_{L_2} has linear relations, then I_L does not have a pure resolution since we have at least two distinct shifts in degree 1 for I_L. Therefore, in the sequel, we may assume that L is a simple lattice.

If L is not cyclic, then, by removing appropriate rows and columns of L, we get an induced sublattice of L of the form displayed in Figure 28.
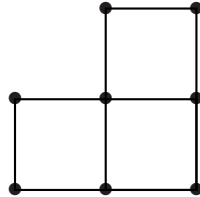

FIGURE 28

The resolution of $I_{L'}$, where L' is the lattice of Figure 28, is the following:

$$0 \to S(-5) \to S(-3)^5 \to S(-2)^5 \to I_{L'} \to 0.$$

Hence, if L is not cyclic, then $\beta_{13}(I_L) \neq 0$. This implies that if I_L has a pure resolution, then I_L must be linearly related, hence the lattice L has the shape indicated in Theorem 3.12.

Now we state the main result of this subsection.

Theorem 3.15. *Let L be a simple planar distributive lattice. Then I_L has a pure resolution if and only if one of the following conditions hold:*

(i) $L = \mathcal{I}(P)$ *where P consists of a chain and an isolated vertex;*
(ii) L *is a cyclic lattice;*
(iii) L *is isomorphic either to the lattice displayed in Figure 28 or to that one displayed in Figure 29.*
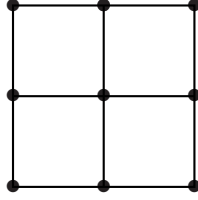
FIGURE 29

Proof. The "if" part is already clear since one may check with a computer that the idea of the lattice pictured in Figure 29 has a pure resolution. For the converse, let us consider a simple planar distributive lattice $L \subset [(0,0),(m,n)]$ such that I_L has a linear resolution.

If $m = 1$ or $n = 1$, then L satisfies condition (i).

Let $m, n \geq 2$ and assume that L is not cyclic. We have to show that L satisfies condition (iii). By the arguments given before the theorem, we know that L must satisfy the conditions of Theorem 3.12.

If L is the whole interval $[(0,0),(m,n)]$ and $m \geq 3$ or $n \geq 3$, then we may obtain an induced sublattice isomorphic to the lattice displayed in Figure 30 which has the property that I_L does not have a pure resolution. This check can be done by using a computer.
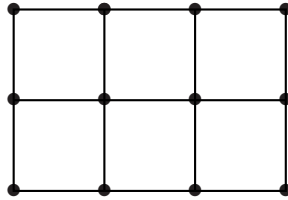
FIGURE 30

Therefore, in this case we get $m = n = 2$ and L is the lattice given in Figure 29.

Let us now suppose that L does not contain the vertex $(0, n)$. Then, by Theorem 3.12, L contains the vertex $(1, n - 1)$. If $m \geq 3$ or $n \geq 3$, then, by removing

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suitable rows and columns of L we get an induced sublattice isomorphic to one of those pictured in Figure 31.

(a)   (b)

Figure 31

None of the lattice displayed above has an ideal with pure resolution as one may check with the computer. Hence, I_L itself does not have a pure resolution. Therefore, in this last case, if I_L has a pure resolution, then L must be isomorphic to the lattice displayed in Figure 29.

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