Divisors on graphs, orientations, syzygies, and system reliability

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Algebraic and Combinatorial Applications of Toric Ideals
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Motivation:

These ideals appear in several different contexts. One can prove many numerical facts about one ideal by looking 'instead' at another ideal in this family. We can pick our favorite ideal to study its numerical invariants, and then translate them to the original setting.

- Graphic arrangements (Greene-Zaslavsky 1983, Novik-Postnikov-Sturmfels 2002, M-Shokrieh 2014)
- Parking functions (Postnikov-Shapiro 2004)
- Theory of divisors on graphs (Baker-Norine 2007, Manjunath-Sturmfels 2012, M-Shokrieh 2013)
- Deformation of Quasisymmetry Models (Kateri-M-Sturmfels 2014)
- The theory of system reliability (M. 2014)
- Percolation theory (M-Sáenz-Wynn, 2014)

Lattice ideals:

- G is a simple graph with $n = |V(G)|$
- Laplacian matrix of G: The symmetric $n \times n$ matrix with

  $a_{ij} = |\{\text{edges between } v_i \text{ and } v_j\}|$
  $a_{ii} = -\deg(v_i)$.
- $L(G) \subset Z^n$: generated by the columns of the Laplacian matrix.
- $S = K[x_i : v_i \in V(G)]$
- Lattice ideal:

$$I_G \;=\; \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \mathbf{u}, \mathbf{v} \in N^n, \mathbf{u} - \mathbf{v} \in L(G) \rangle.$$

Questions:

Describe the algebraic invariants (a minimal free resolution) of I_G in combinatorial terms of graph.

More precisely:

Give a polyhedral complex minimally resolving the resolution of I_G.

For example:

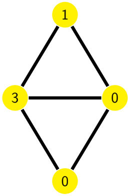
- Betti numbers
- regularity: $|E(G)| - |V(G)| + 1$
- Multiplicity (The leading coefficient of Hilbert polynomial): Number of spanning trees
- h-vector as an evaluation of Tutte polynomial T(1,y)
- The CW-complex resolving the minimal free resolution of I_G.

Divisors on graphs

- G is a simple graph
- $Div(G)$: free abelian group generated by $V(G)$

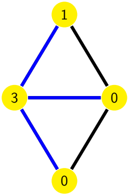
$$D = \sum_{v \in V(G)} a_v(v),$$

$D(v) := a_v \in \mathbb{Z}$.



Chip-firing game:

- initial configuration: assign an integer number of dollars to each vertex, D
- move: consists of a vertex v either borrowing one dollar from each of its neighbors or giving one dollar to each of its neighbors.
- $D \sim D'$: there is a sequence of moves taking D to D' in the chip-firing game.



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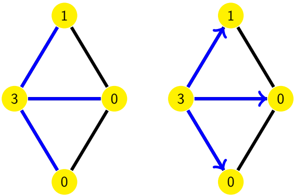

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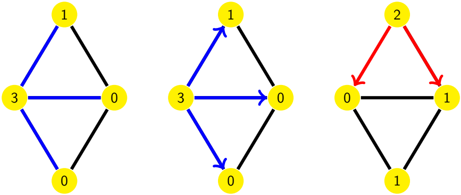

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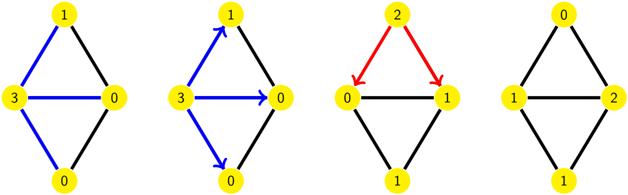


The toppling ideal I_G

- $S = K[x_i : i \in V(G)]$
- $I_G := \langle \mathbf{x}^{D_1} - \mathbf{x}^{D_2} : D_1 \sim D_2 \text{ and } D_1, D_2 \geq 0 \rangle$
- $M_G := \mathrm{in}_{\mathrm{revlex}}(I_G)$ with respect to $x_1 > \cdots > x_n$.

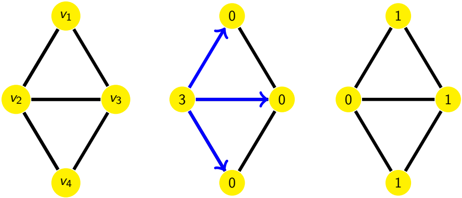
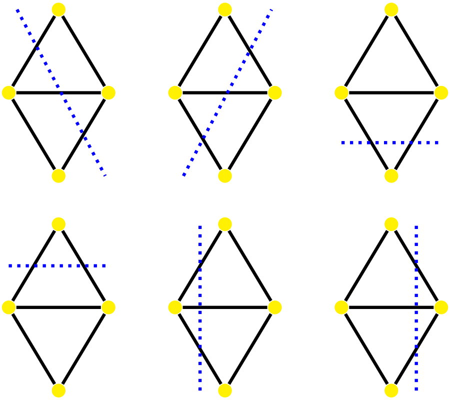


Figure: $x_2^3 - x_1 x_3 x_4$

Connected 2-partitions (Minimal generating set of I_G)



Binomial associated to an 2-acyclic orientation

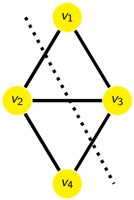


Figure: $x_1 x_3^2 - x_2^2 x_4$

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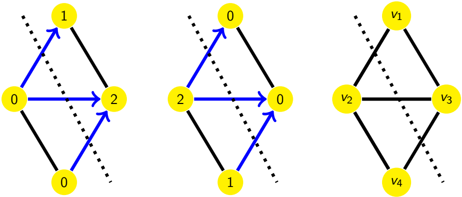

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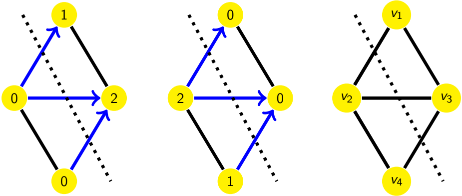

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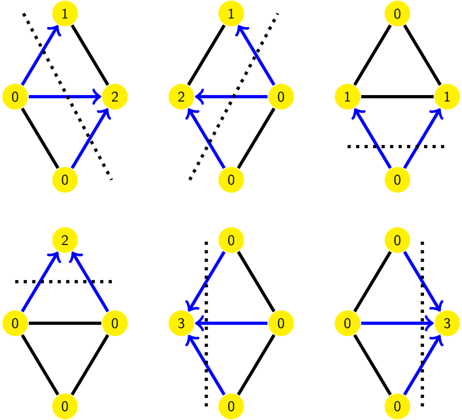
2-acyclic orientations (Minimal generating set of M_G)



Minimal free resolution of I_G?

Theorem (Novik-Postnikov-Sturmfels 2002, M-Shokrieh 2013)

There is a one-to-one correspondence between:

(1) $(k-2)^{\mathrm{th}}$ syzygies of I_G and M_G (its distinguished initial ideal)
(2) k-connected flags of G with unique source
(3) k-acyclic orientations of G with unique source
(4) maximal q-reduced divisors on the partition graphs
(5) k-dimensional bounded regions of the graphical arrangement.

Theorem (Postnikov-Shapiro 2004)

For the complete graph K_n, $\beta_{k-2}(M_{K_n}) = (k-1)!S(n,k)$, where $S(n,k)$ denotes the Stirling number of the second kind (i.e. the number of ways to partition a set of n elements into k nonempty subsets).

Hyperplane arrangements

Definition

- Corresponding to each edge ij of G with $i < j$

$$H_{ij} := \{v \in \mathbb{R}^n : \; h_{ij}(v) = 0 \;\; \textit{for} \;\; h_{ij}(v) := v_i - v_j\}.$$

- The graphical hyperplane arrangement of G is

$$\mathcal{A}_G := \{H_{ij} : \; ij \in E(G) \;\; \text{and} \;\; i < j\}.$$

- $\mathcal{H}_G$: The restriction of $\mathcal{A}_G$ to

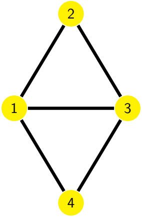
$$H_q := \{v \in \mathbb{R}^n : \; v_n = 0 \;\; \text{and} \;\; v_1 + \cdots + v_{n-1} = 1\} \; .$$

Example

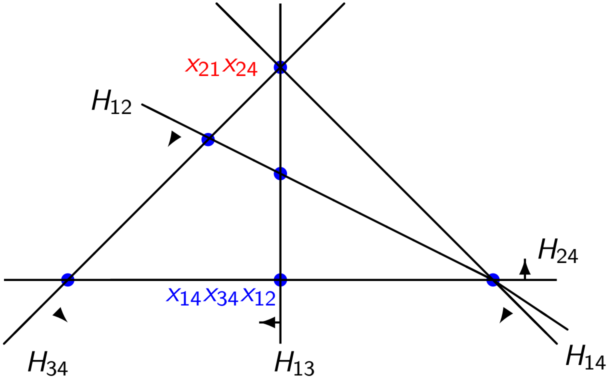
$\mathcal{H}_G$ is the restriction of

$$\mathcal{A}_G := \{H_{12}, H_{24}, H_{34}, H_{14}, H_{13}\}$$

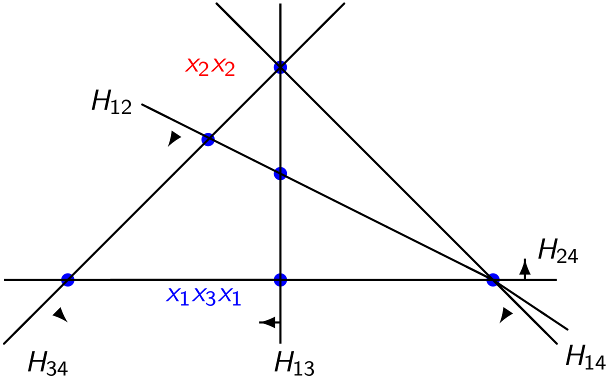
to $H_q = \{v \in \mathbb{R}^4 : \ v_4 = 0 \ \text{ and } \ v_1 + v_2 + v_3 = 1\}$.



Oriented matroid ideal:



Relabeling→initial ideal of the toppling ideal:



Minimal free resolution of O_G and M_G?

Theorem (Novik-Postnikov-Sturmfels 2002, M-Shokrieh 2012)

The bounded complex of the graphical arrangementre supports a minimal free resolution for the oriented matroid ideal, and the initial ideal of the toppling ideal (studied by Postnikov-Shapiro 2004).

$$0 \to R^4 \to R^9 \to R^6 \to R$$

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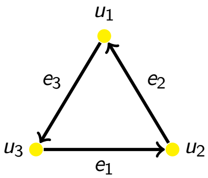
Proof: Relabeling makes sense from Algebraic point of view!

Polyhedral complex: Convex geometry and potential theory.

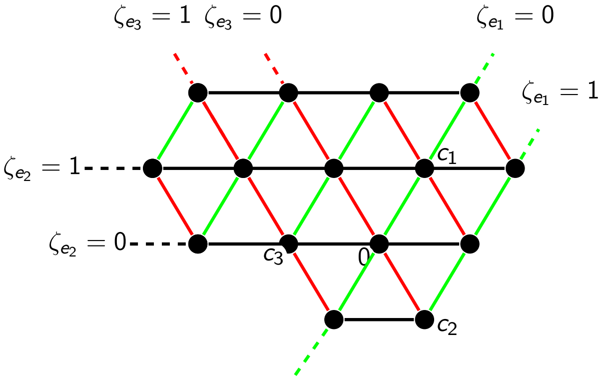
Graph K_3 and a fixed orientation:



Remember the columns of the Laplacian matrix:
$(-2, 1, 1),\ (1, -2, 1),\ (1, 1, -2)$.

$c_1 = -2(u_1) + (u_2) + (u_3)$,
$c_2 = (u_1) - 2(u_2) + (u_3)$, $c_3 = (u_1) + (u_2) - 2(u_3)$.

The lattice $Prin(K_3) \subseteq \mathbb{R}^3$



Minimal free resolution of J_G and I_G?

Theorem (M-Shokrieh 2013)

The quotient cell complex $\mathrm{Del}(\mathrm{Prin(G)}/\mathrm{Prin(G)})$ supports a $\mathrm{Pic(G)}$-graded minimal free resolution for I_G.

After drawing the Delaunay decomposition of $(Prin(G), \langle ., . \rangle_{en})$, we will see lots of hyperplanes corresponding to the edges of the graph!

How to read the resolution of the Lawrence ideal, and the Toppling ideal from constructed complex?

- Pick a fundamental domain of 'Delaunay Decomposition' of $(Prin(G), \langle ., . \rangle_{en})$
- Label the faces with Laurent monomials; the vertices take their labels from the corresponding cuts, and the face $F = \{v_1, \ldots, v_k\}$ are labeled by the lcm of the labels of v_i's.

Fundamental domain for K_3

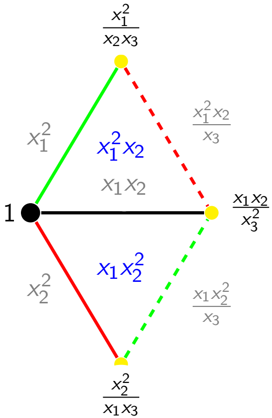


Figure: A choice of fundamental domain with labels

Thank you!