A canonical form of a factor of monomial ideals and related algorithms

Adrian Popescu

TU Kaiserslautern

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Contents

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- I is **in the canonical form with respect to** x_n if it is of type $(1, 2, \ldots, s)$ for some $s \in \mathbb{N}$.

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- I is **in the canonical form with respect to** x_n if it is of type $(1, 2, \ldots, s)$ for some $s \in \mathbb{N}$.
- We simply say that I is **the canonical form** if it is in the canonical form with respect to all variables $x_1, \ldots, x_n$.

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Example

Consider $R = \mathbb{Q}[x, y]$ and the monomial ideal $I = (x^6, x^3y^7)$.
Note that I is of type (7) wrt y (hence $y^7 \mapsto y$) and of type $(3, 6)$ wrt x (therefore $x^3 \mapsto x$ and $x^6 \mapsto x^2$).
Then the canonical form of I is $\bar{I} = (x^2, xy)$.

A canonical form of a factor of monomial ideals

Definition (factor case)

Let $J \subsetneq I \subset R$ be two monomial ideals. We say that I/J is *of type* $(k_1, \ldots, k_s)$ *with respect to* x_n if $x_n^{k_i}$ are all the powers of x_n which enter in a monomial of $G(I) \cup G(J)$ for $i \in \{1, \ldots, s\}$ and $1 \leq k_1 < \ldots < k_s$.
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Example

Consider $R = \mathbb{Q}[x, y, z]$, $I = (x^{10}y^5, x^4yz^7, z^7y^3)$ and $J = (x^{10}y^{20}z^2, x^3y^4z^{13}, x^9y^2z^7)$.
The canonical form of I/J is $\overline{I/J} = \dfrac{(x^4y^5, x^2yz^2, y^3z^2)}{(x^4y^6z, xy^4z^3, x^3y^2z^2)}$.

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Example

Let $R = \mathbb{Q}[x, y]$, $I = (x^4, y^{10}, x^2y^7)$ and $J = (x^{20}, y^{30})$. The canonical form of I is $\overline{I} = (x^2, y^2, xy)$ and the canonical form of J is $\overline{J} = (x, y)$. Then $\overline{J} \not\subset \overline{I}$.

But the canonical form of the factor I/J is $\overline{I/J} = \dfrac{(x^2, y^2, xy)}{(x^3, y^3)}$.

The Stanley Depth

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1 \dots, x_n]$. On R consider the $\mathbb{Z}^n-$grading in which each x_i has degree the $i-$th vector of the canonical basis.

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A **Stanley decomposition** of a $\mathbb{Z}^n-$graded $R-$module M is a finite family

$$\mathcal{D} = (R_i, u_i)_{i \in I}$$

in which u_i are homogeneous elements of M and R_i is a $\mathbb{Z}^n-$graded $K-$algebra retract of R for each $i \in I$ such that $R_i \cap Ann(u_i) = 0$ and

$$M = \bigoplus_{i \in I} R_i u_i$$

as a graded $\mathbb{K}-$vector space.

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The **Stanley depth** of $\mathcal{D}$ denoted by $sdepth_n\, \mathcal{D}$ is the depth of the $R-$module $\bigoplus_{i \in I} R_i u_i$. The **Stanley depth** of M is defined as

$$sdepth_n(M) = \max\{sdepth_n\, \mathcal{D} \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

Depth and Stanley Depth of $\overline{I/J}$

Conjecture (Stanley, 1982)

$$sdepth\ M \geq depth\ M.$$

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Theorem

Let I, J be monomial ideals in R and $\overline{I/J}$ the canonical form of the factor I/J. Then

$$sdepth_n I/J = sdepth_n \overline{I/J}.$$

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This follows easily from [B. Ichim, L. Katthän and J. J. Moyano-Fernández, *The behaviour of Stanley depth under polarization*, Proposition 5.1]

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This was proved in [A. Popescu, *Depth and Stanley Depth of the Canonical Form of a factor of monomial ideals*, Theorem 2]

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Theorem

The Stanley Conjecture holds for a factor of monomial ideals I/J if and only if it holds for its canonical form $\overline{I/J}$.

Timings on existing algorithms using the canonical form

We will use the canonical form in G. Rinaldo's algorithm introduced in *An algorithm to compute the Stanley depth*.

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One can also see some small improvements in the *depth* computation algorithm used in SINGULAR.

The Hilbert Depth

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After Bruns-Krattenthaler-Uliczka, a **Hilbert decomposition** of a $\mathbb{Z}-$graded $R-$module M is a finite family

$$\mathcal{H} = (R_i, s_i)_{i \in I}$$

in which $s_i \in \mathbb{Z}$ and R_i is a $\mathbb{Z}-$graded $K-$algebra retract of R for each $i \in I$ such that

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$$hdepth_1(M) = \max\{hdepth_1\, \mathcal{H} \mid \mathcal{H} \text{ is a Hilbert decomposition of } M\}.$$

Difference between the Hilbert and Stanley depth

Example (Bruns, Krattenthaler and Uliczka, Stanley Decompositions and Hilbert Depth in the Koszul Complex)

Consider $R := \mathbb{K}[x, y]$ and $M := \mathbb{K} \oplus y\frac{R}{x} \oplus yR$. We get $sdepth_1\ M = sdepth_2\ M = 0$ and $hdepth_1\ M = 2$.

An algorithm to compute the Hilbert Depth

Definition

A Laurent series in $\mathbb{Z}[\![t, t^{-1}]\!]$ is called **positive** if it has only nonnegative coefficients.

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Theorem (J. Uliczka, *Remarks on Hilbert series of graded modules over polynomial rings*)

$hdepth_1(M) = \max\{e \mid (1-t)^e \cdot HP_M(t) \text{ is positive}\}$, *where* $HP_M(t)$ *is the Hilbert–Poincaré series of* M.

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∘Based on this Theorem, one can construct an algorithm which computes the $hdepth_1$ of a module M.

∘The only difficulty that arises is:

Question

How many coefficients of the infinite Laurent series one have to check in order to say that it is positive?

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An easy trick was found in [A. Popescu, *An Algorithm to compute the Hilbert Depth*, Journal of Symbolic Computation Volume 66, January-February 2015, Pages 1-7].

The main idea of the algorithm and examples

Example

Consider $I = x \cap (y_1, \ldots, y_5)$ for which $\dim\ I = 5$ and $G(t) = 1 + t - 4t^2 + 6t^3 - 4t^4 + t^5$.

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Consider $G(t) = 2 - 3t - 2t^2 + 2t^3 + 4t^4$.

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$$G(t)/(1-t)^8 = 1 + 3t + 3t^2 - 4t^3 - 18t^4 - 18t^5 + 66t^6 + \ldots$$
$$G(t)/(1-t)^9 = 1 + 4t + 7t^2 + 3t^3 - 15t^4 - 33t^5 + 33t^6 + \ldots$$
$$G(t)/(1-t)^{10} = 1 + 5t + 12t^2 + 15t^3 - 33t^5 + \ldots$$
$$G(t)/(1-t)^{11} = 1 + 6t + 18t^2 + 33t^3 + 33t^4 + \ldots$$

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For $n = 6$, $hdepth_1(R \oplus m) > hdepth_1 m$, which is a sign that in this case $sdepth_n(R \oplus m) > sdepth_n m$ and so Herzog's question could have a negative answer for $n = 6$.

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This is indeed the case as it was shown later by B. Ichim and A. Zarojanu in *An algorithm for computing the multigraded Hilbert depth of a module*.

Thank you for your attention.