Polyomino Ideals

Ayesha Asloob Qureshi

Abdus Salam International Centre of Theoretical Physics, Trieste

The 22nd National School on Algebra, 2012
Romania, September 1-5, 2014

- A. A. Qureshi, Ideals generated by 2-minors, collections of cells and stack polyominoes, Journal of Algebra, **357**, 279–303, (2012).
- J. Herzog, A. A. Qureshi, A. Shikama, Gröbner basis of balanced polyominoes, to appear in Math. Nachr.

Polyominoes

Polyominoes are, roughly speaking, plane figures obtained by joining squares of equal size (cells) edge to edge. Their appearance origins in recreational mathematics but also has been a subject of many combinatorial investigations including tiling problems.

Polyominoes

A connection of polyominoes to commutative algebra has been established first in (-, 2012) by assigning to each polyomino its ideal of inner minors (also called Polyomino ideals).

This class of ideals widely generalizes the ideal of 2-minors of a matrix of indeterminates, and even that of the ideal of 2-minors of two-sided ladders. It also includes the meet-join ideals (Hibi ideals) of planar distributive lattices.

Polyominoes

For any integer $1 \leq t \leq \min\{m, n\}$, the ideal generated by all t-minors of X is well understood and discussed in several papers, for example, [Hochster, Eagon (1971)] and [Bruns, Vetter (1988)], and more generally the ideals generated by all t-minors of a one and two sided ladders, see for example [Conca (1995)].

Polyominoes

For any integer $1 \leq t \leq \min\{m, n\}$, the ideal generated by all t-minors of X is well understood and discussed in several papers, for example, [Hochster, Eagon (1971)] and [Bruns, Vetter (1988)], and more generally the ideals generated by all t-minors of a one and two sided ladders, see for example [Conca (1995)].

Motivated by applications in algebraic statistics, ideals generated by even more general sets of minors have been investigated, including ideals generated by adjacent 2-minors, see [Hoşten, Sullivant (2004)], [Hibi, Ohsugi (2006)] and [Herzog, Hibi (2010)], and ideals generated by an arbitrary set of 2-minors in a $2 \times n$-matrix [Herzog, Hibi, Hreinsdóttir, Kahle, Rauh, (2010)].

Typically one determines for such ideals their Gröbner bases, determines their resolution and computes their regularity, checks whether the rings defined by them are normal, Cohen-Macaulay or Gorenstein.

Polyominoes

In order to define the polyominoes and polyomino ideals, we introduce some terminology.

Polyominoes

In order to define the polyominoes and polyomino ideals, we introduce some terminology.

We denote by $<$ the natural partial order on $\mathbb{N}^2$, i.e. $(i,j) \leq (k,l)$ if and only if $i \leq k$ and $j \leq l$.

Polyominoes

In order to define the polyominoes and polyomino ideals, we introduce some terminology.

We denote by $<$ the natural partial order on $\mathbb{N}^2$, i.e. $(i, j) \leq (k, l)$ if and only if $i \leq k$ and $j \leq l$.
Let $a = (i, j)$ and $b = (k, l)$ in $\mathbb{N}^2$. Then

(1) the set $[a, b] = \{c \in \mathbb{N}^2 : a \leq c \leq b\}$ is called an interval,

(2) the interval of the form $C = [a, a + (1, 1)]$ is called a cell. (with lower left corner a),

(3) the elements of C are called the *vertices* of C, and the sets $\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}$ and $\{a + (0, 1), a + (1, 1)\}$ the *edges* of C.

Polyominoes

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let C and D be two cells of $\mathcal{P}$. Then C and D are said to be *connected*, if there is a sequence of cells $C : C_1, \ldots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m-1$. The collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected.

Polyominoes

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let C and D be two cells of $\mathcal{P}$. Then C and D are said to be *connected*, if there is a sequence of cells $C : C_1, \ldots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m-1$. The collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected.

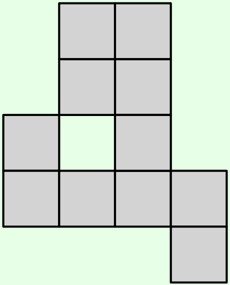


Figure : A polyomino

Polyomino ideals

Let $\mathcal{P}$ be a polyomino, and let K be a field. We denote by S the polynomial over K with variables x_{ij} with $(i,j) \in V(\mathcal{P})$. A 2-minor $x_{ij}x_{kl} - x_{il}x_{kj} \in S$ is called an *inner minor* of $\mathcal{P}$ if all the cells $[(r,s),(r+1,s+1)]$ with $i \leq r \leq k-1$ and $j \leq s \leq l-1$ belong to $\mathcal{P}$. In that case the interval $[(i,j),(k,l)]$ is called an *inner interval* of $\mathcal{P}$. The ideal $I_{\mathcal{P}} \subset S$ generated by all inner minors of $\mathcal{P}$ is called the *polyomino ideal* of $\mathcal{P}$. We also set $K[\mathcal{P}] = S/I_{\mathcal{P}}$.

Polyomino ideals

Our main objective is to classify all polyominoes such that associated polyomino ideals are prime.

Convex Polyominoes

As a first step in this direction, we studied the *convex* polyominoes.

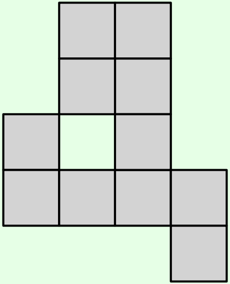


Figure : Not covnex

i

Convex Polyominoes

Theorem

Let $\mathcal{P}$ be a convex collection of cells. Then $K[\mathcal{P}]$ is a normal Cohen–Macaulay domain of dimension $|V(\mathcal{P})| - |\mathcal{P}|$.

Simple Polyominoes

For simple polyominoes, in (-, 2012), it was conjectured that $I_{\mathcal{P}}$ is a prime ideal.

Simple Polyominoes

For simple polyominoes, in (-, 2012), it was conjectured that $I_{\mathcal{P}}$ is a prime ideal.

Let $\mathcal{P}$ be a polyomino and let $[a, b]$ an interval with the property that $\mathcal{P} \subset [a, b]$. A polyomino $\mathcal{P}$ is called *simple*, if for any cell C not belonging to $\mathcal{P}$ there exists a path $C = C_1, C_2, \ldots, C_m = D$ with $C_i \notin \mathcal{P}$ for $i = 1, \ldots, m$ and such that D is not a cell of $[a, b]$.
Roughly speaking, a polyomono is called simple if it has no holes.

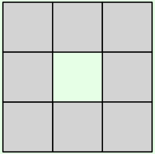


Figure : Not simple

Admissibe labeling

Let $\mathcal{P}$ be a polyomino. An interval $[a, b]$ with $a = (i, j)$ and $b = (k, l)$ is called a *horizontal edge interval* of $\mathcal{P}$ if $j = l$ and the sets $\{r, r+1\}$ for $r = i, \ldots, k-1$ are edges of cells of $\mathcal{P}$. Similarly one defines vertical edge intervals of $\mathcal{P}$. We call, an integer value function $\alpha\colon V(\mathcal{P}) \to \mathbb{Z}$ is called *admissible*, if for all maximal horizontal or vertical edge intervals $\mathcal{I}$ of $\mathcal{P}$ one has

$$\sum_{a \in \mathcal{I}} \alpha(a) = 0.$$

Admissibe labeling

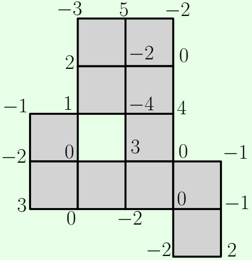


Figure : An admissible labeling

Balanced polyominoes

Given an admissible labeling α, we define the binomial

$$f_\alpha = \prod_{\substack{a \in V(\mathcal{P}) \\ \alpha(a) > 0}} x_a^{\alpha(a)} - \prod_{\substack{a \in V(\mathcal{P}) \\ \alpha(a) < 0}} x_a^{-\alpha(a)},$$

Balanced polyominoes

Given an admissible labeling α, we define the binomial

$$f_\alpha = \prod_{\substack{a \in V(\mathcal{P}) \\ \alpha(a) > 0}} x_a^{\alpha(a)} - \prod_{\substack{a \in V(\mathcal{P}) \\ \alpha(a) < 0}} x_a^{-\alpha(a)},$$

Let $J_\mathcal{P}$ be the ideal generated by the binomials f_α where α is an admissible labeling of $\mathcal{P}$. It is obvious that $I_\mathcal{P} \subset J_\mathcal{P}$. We call a polyomino *balanced* if for any admissible labeling α, the binomial $f_\alpha \in I_\mathcal{P}$. This is the case if and only if $I_\mathcal{P} = J_\mathcal{P}$.

Balanced polyominoes

Consider the free abelian group $G = \bigoplus_{(i,j)\in V(\mathcal{P})} \mathbb{Z}e_{ij}$ with basis elements e_{ij}. To any cell $C = [(i,j),(i+1,j+1)]$ of $\mathcal{P}$ we attach the element $b_C = e_{ij} + e_{i+1,j+1} - e_{i+1,j} - e_{i,j+1}$ in G and let $\Lambda \subset G$ be the lattice spanned by these elements.

Balanced polyominoes

Lemma

The elements b_C form a K-basis of Λ and hence $\operatorname{rank}_{\mathbb{Z}} \Lambda = |\mathcal{P}|$. *Moreover, Λ is saturated. In other words, G/Λ is torsionfree.*

Balanced polyominoes

The lattice ideal I_Λ attached to the lattice Λ is the ideal generated by all binomials

$$f_v = \prod_{\substack{a \in V(\mathcal{P}) \\ v_a > 0}} x_a^{v_a} - \prod_{\substack{a \in V(\mathcal{P}) \\ v_a < 0}} x_a^{-v_a}$$

with $v \in \Lambda$.

Balanced polyominoes

The lattice ideal I_Λ attached to the lattice Λ is the ideal generated by all binomials

$$f_v = \prod_{\substack{a \in V(\mathcal{P}) \\ v_a > 0}} x_a^{v_a} - \prod_{\substack{a \in V(\mathcal{P}) \\ v_a < 0}} x_a^{-v_a}$$

with $v \in \Lambda$.

Proposition

Let $\mathcal{P}$ be a balanced polyomino. Then $I_\mathcal{P} = I_\Lambda$ and has height $|\mathcal{P}|$.

Gröbner basis of balanced polyominoes

The primitive binomials in $\mathcal{P}$ are determined by cycles.

Gröbner basis of balanced polyominoes

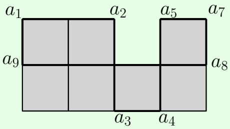
The primitive binomials in $\mathcal{P}$ are determined by cycles.

A sequence of vertices $\mathcal{C} = a_1, a_2, \ldots, a_m$ in $V(\mathcal{P})$ with $a_m = a_1$ and such that $a_i \neq a_j$ for all $1 \leq i < j \leq m-1$ is a called a *cycle* in $\mathcal{P}$ if the following conditions hold:

(i) $[a_i, a_{i+1}]$ is a horizonal or vertical edge interval of $\mathcal{P}$ for all $i = 1, \ldots, m-1$;

(ii) for $i = 1, \ldots, m$ one has: if $[a_i, a_{i+1}]$ is a horizonal interval of $\mathcal{P}$, then $[a_{i+1}, a_{i+2}]$ is a vertical edge interval of $\mathcal{P}$ and vice versa. Here, $a_{m+1} = a_2$.

Gröbner basis of balanced polyominoes



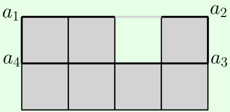


Figure : A cycle and a non-cycle in $\mathcal{P}$

Gröbner basis of balanced polyominoes

It follows immediately from the definition of a cycle that $m-1$ is even. Given a cycle $\mathcal{C}$, we attach to $\mathcal{C}$ the binomial

$$f_{\mathcal{C}} = \prod_{i=1}^{(m-1)/2} x_{a_{2i-1}} - \prod_{i=1}^{(m-1)/2} x_{a_{2i}}$$

Gröbner basis of balanced polyominoes

Theorem

Let $\mathcal{P}$ be a balanced polyomino.

(a) *Let $\mathcal{C}$ be a cycle in $\mathcal{P}$. Then $f_{\mathcal{C}} \in I_{\mathcal{P}}$.*

(b) *Let $f \in I_{\mathcal{P}}$ be a primitive binomial. Then there exists a cycle $\mathcal{C}$ in $\mathcal{P}$ such that each maximal interval of $\mathcal{P}$ contains at most two vertices of $\mathcal{C}$ and $f = \pm f_{\mathcal{C}}$.*

Gröbner basis of balanced polyominoes

Theorem

Let $\mathcal{P}$ be a balanced polyomino.

(a) Let $\mathcal{C}$ be a cycle in $\mathcal{P}$. Then $f_{\mathcal{C}} \in I_{\mathcal{P}}$.

(b) Let $f \in I_{\mathcal{P}}$ be a primitive binomial. Then there exists a cycle $\mathcal{C}$ in $\mathcal{P}$ such that each maximal interval of $\mathcal{P}$ contains at most two vertices of $\mathcal{C}$ and $f = \pm f_{\mathcal{C}}$.

Corollary

Let $\mathcal{P}$ be a balanced polyomino. Then $I_{\mathcal{P}}$ admits a squarefree initial ideal for any monomial order.

Classes of balanced polyominoes

The polyomino $\mathcal{P}$ is called *tree-like* if each subpolyomino of $\mathcal{P}$ has a leaf.

Classes of balanced polyominoes

The polyomino $\mathcal{P}$ is called *tree-like* if each subpolyomino of $\mathcal{P}$ has a leaf.

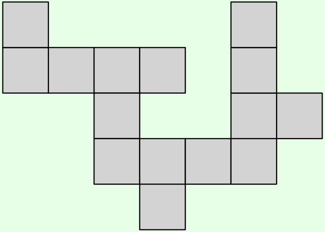


Figure : A tree-like polyomino

Classes of Balanced polyominoes

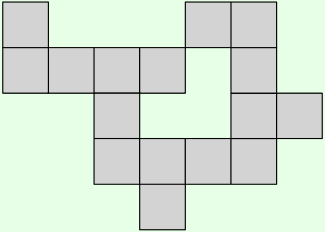


Figure : Not a tree-like polyomino

Classes of balanced polyominoes

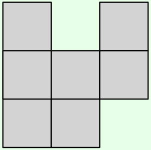


Figure : Column convex but not row convex

Classes of balanced polyominoes

Theorem

Let $\mathcal{P}$ be a row or column convex, or a tree-like polyomino. Then $\mathcal{P}$ is balanced and simple.

Classes of Balanced polyominoes

Theorem

A polyomino is simple if and only if it is balanced.

Polyominoes

Very recently those convex polyominoes have been classified in [Ene, Herzog, Hibi (2014)] whose ideal of inner minors is linearly related or has a linear resolution. For some special polyominoes also the regularity of the ideal of inner minors is studied by [Ene, Rauf, - (2013)].

Questions

What is the complete characterization of prime polyominoes.

Questions

What is the complete characterization of prime polyominoes.

Can we find another class of prime polyominoes different than simple polyominoes?

Questions

What is the complete characterization of prime polyominoes.

Can we find another class of prime polyominoes different than simple polyominoes?

What can we say about the ideal generated by inner t-minors of a polyomino?

Questions

What is the complete characterization of prime polyominoes.

Can we find another class of prime polyominoes different than simple polyominoes?

What can we say about the ideal generated by inner t-minors of a polyomino?

Does there exist a monomial order such that polyomino ideals (or any particular class of polyominoe ideals) have quadratic Gröbner basis?

Questions

What is the complete characterization of prime polyominoes.

Can we find another class of prime polyominoes different than simple polyominoes?

What can we say about the ideal generated by inner t-minors of a polyomino?

Does there exist a monomial order such that polyomino ideals (or any particular class of polyominoe ideals) have quadratic Gröbner basis?

When convex polyominoes are Gorenstien?

Thank you!

References

- A. Conca, Ladder determinantal rings, J. Pure Appl. Algebra **98**, 119–134 (1995)
- V. Ene, J. Herzog, T. Hibi, Linearly related polyominoes, arXiv:1403.4349 [math.AC]
- J. Herzog, T. Hibi, F. Hreinsdóttir, T. Kahle, J. Rauh. Binomial edge ideals and conditional independence statements, Adv. Appl. Math. **45**, 317–333 (2010)
- J. Herzog, T. Hibi, Ideals generated by adjacent 2–minors, Journal of Commutative Algebra **4**, 525–549 (2012)
- J. Herzog, S. Madani, The coordinate ring of a simple polyomino, arXiv:1408.4275v1, (2014)

References

- J. Herzog, A. A. Qureshi, A. Shikama, Gröbner basis of balanced polyominoes, to appear in Math. Nachr.
- M. Hochster and J.A. Eagon, Cohen–Macaulay rings, invariant theory and the generic perfection of determinantal loci. Amer.J.Math. **93**, 1020–1058 (1971)
- S. Hoşten, S. Sullivant, Ideals of adjacent minors, J. Algebra **277** , 615–642 (2004)
- H. Ohsugi, T. Hibi, Toric ideals generated by quadratic binomials. J. Algebra **218**, 509–527 (1999)
- H. Ohsugi, T. Hibi, Special simplices and gorenstein toric rings. J. Combinatorial Theory Series A **113**, (2006)
- J. Shapiro, S. Hoşten, Primary decomposition of lattice basis ideals. J.Symbolic Computation **29**, 625–639 (2000)

References

- B. Sturmfels, Gröbner Bases and Convex Polytopes, Amer. Math. Soc., Providence, RI, (1995)
- B. Sturmfels, D. Eisenbud, Binomial ideals, Duke Math. J. **84** , 1–45 (1996)
- U. Vetter, W. Bruns, Determinantal rings, Lecture Notes in Mathematics, Springer, (1988)
- R. Villarreal, Monomial Algebras. Marcel Dekker
- A. A. Qureshi, Ideals generated by 2-minors, collections of cells and stack polyominoes, Journal of Algebra, **357**, 279–303, (2012).