LECTURE ON MATROID TORIC IDEALS

JORGE RAMÍREZ ALFONSÍN

Abstract. In this notes, we give a basic introduction of the study of matroids toric ideals and some applications.

1. Introduction

Let M be a matroid on a finite ground set $E = \{1, \ldots, n\}$, we denote by $\mathcal{B}$ the set of bases of M. Let k be an arbitrary field and consider $k[x_1, \ldots, x_n]$ a polynomial ring over k. For each base $B \in \mathcal{B}$, we introduce a variable y_B and we denote by R the polynomial ring in the variables y_B, i.e., $R := k[y_B \,|\, B \in \mathcal{B}]$. A *binomial* in R is a difference of two monomials, an ideal generated by binomials is called a *binomial ideal.*

We consider the homomorphism of k-algebras $\varphi : R \longrightarrow k[x_1, \ldots, x_n]$ induced by

$$y_B \mapsto \prod_{i \in B} x_i.$$

The image of φ is a standard graded k-algebra, which is called the *bases monomial ring of the matroid* M and it is denoted by S_M. By a result due to White [17, Theorem 5], S_M has Krull dimension $\dim(S_M) = n - c + 1$, where c is the number of connected components of M. The kernel of φ, which is the presentation ideal of S_M, is called the *toric ideal of* M and is denoted by I_M.

It is well known that I_M is a prime, binomial and homogeneous ideal, see, e.g., [15]. Since $R/I_M \simeq S_M$, it follows that the height of I_M is $\mathrm{ht}(I_M) = |\mathcal{B}| - \dim(S_M)$. Let b be the number of bases of M, given $\mathbf{u} \in \mathbb{Z}^b$ define $\mathbf{u}_+$ (resp. $\mathbf{u}_-$ to be $\mathbf{u}$ (resp. $-\mathbf{u}$) with negative coordinates replaced by zeros; we then have $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$. From the point of view of Sturmfels [16] toric ideals are generated by binomials $\mathbf{y}^{\mathbf{u}_+} - \mathbf{y}^{\mathbf{u}_-}$, where $\mathbf{u}_+$ runs over integer vectors in the kernel of an integer matrix. For the toric ideal I_M the integer matrix is the $m \times b$ matrix whose columns are the zero-one incidence vectors of the bases of M.

Let M be a matroid on the ground set $E = \{1, \ldots, n\}$ and rank $r \geq 2$. Let $\mathcal{B}$ denote the set of bases of M. By definition $\mathcal{B}$ is not empty and satisfies the following *exchange axiom*:

> For every $B_1, B_2 \in \mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$.

Brualdi proved in [5] that the exchange axiom is equivalent to the *symmetric exchange axiom*:

> For every B_1, B_2 in $\mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that both $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$ and $(B_2 \cup \{e\}) \setminus \{f\} \in \mathcal{B}$.

This notes were prepared for the 22nd National School on Algebra, Algebraic and Combinatorial Applications of Toric Ideals, Bucharest, Romania, September 2014.

Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$. Then, we say that the quadratic binomial $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ correspond to *symmetric exchange*. It is clear that such binomial belong to the ideal I_M.

Conjecture 1.1. *(White [18]) For every matroid M its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.*

We further observe that for $B_1, \ldots, B_s, D_1, \ldots, D_s \in \mathcal{B}$, the homogeneous binomial $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$ belongs to I_M if and only if $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$ as multisets. Since I_M is a homogeneous binomial ideal, it follows that

$$(1.1)\quad I_M = \big(\{y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s} \,|\, B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s \textit{ as multisets}\}\big).$$

Since every toric ideal is generated by binomials then we can rephrase the above conjecture in the combinatorial language. It asserts that if two sets of bases of a matroid have equal union (as multiset), then one can pass between them by a sequence of symmetric exchanges. In fact this is the original formulation due to White. We immediately see that the conjecture does not depend on the field k.

Blasiak [2] has confirmed the conjecture for graphical matroids, Kashiwaba [10] checked the case of matroids of rank at most 3. Schweig [14] proved the case of *lattice path* matroids which are a subclass of transversal matroids. Recently Bonin [6] confirmed the conjecture for *sparse paving* matroids.

It is also natural to ask whether the following variant of White's conjecture holds (see for instance [7] and [15, Chapter 14])

Conjecture 1.2. *For any matroid M, the toric ideal I_M has a Gröbner basis consisting of quadratics binomials.*

Sturmfels [15, Chapter 14] show that Conjecture 1.2 holds for uniform matroids.
White's conjecture can be posed as two separated conjectures. The following are both still open in their generality and together imply White's conjecture.

Conjecture 1.3. *For any matroid M, the toric ideal I_M is generated by quadratics binomials.*

Conjecture 1.4. *For any matroid M, the quadratic binomials of I_M are in the ideal generated by the binomials $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ such that the pair of bases D_1, D_2 can be obtained from the pair B_1, B_2 by a symmetric exchange.*

2. Blasiak's reduction

In [2] Blasiak showed that the algebraic formulation of White's conjecture is implied by the following combinatorial condition similar to White's original formulation. Let M be a matroid on a ground set E with $|E| = nr(M)$ where $r(M)$ is the rank of M. The *n-base graph* of M, which is denoted by $G_n(M)$, has as its vertex set the set of all sets of n disjoint bases (a set of n bases $\{B_1, \ldots, B_n\}$ of M is disjoint if and only if

$$|E| = \bigcup_{i=1}^{n} B_i.$$

There is an edge between $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ if and only if $B_i = D_j$ for some i, j. Blasiak proved that Conjecture 1.3 is implied by the connectivity of the n-base graphs. Let us first proof the following lemma for a general class of matroids $\mathfrak{C}$ that is closed under deletions and adding parallel elements.

Lemma 2.1. [2] *Let $\mathfrak{C}$ be a collection of matroids that is closed under deletions and adding parallel elements. Suppose that for each $n \geq 3$ and for every matroid M in $\mathfrak{C}$ on a ground set of size $nr(M)$ the n-base graph of M is connected. Then, for every matroid M in $\mathfrak{C}$, I_M is generated by quadratics polynomials.*

Proof. We will prove by induction on n the following statement: for every $M \in \mathfrak{C}$ and every binomial $b \in I_M$ of degree n, b is in the ideal generated by the quadratics of I_M. This will prove the result because as mentioned above I_M is spanned as a k-vector space by binomials. Suppose $n \geq 3$ and $M \in \mathfrak{C}$ on the ground set $\{1, \ldots, m\}$, and b is binomial of degree n in I_M. The binomial b is necessarily of the form $b = \prod_{i=1}^{n} y_{B_i} - \prod_{i=1}^{n} y_{D_i}$ for some bases $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ of M such that the B_i and D_i have the same multiset union. We will show that b is in the ideal generated by the degree $n-1$ binomials of I_M, we will do so by constructing a new matroid M' that depends on the binomial b. By induction the degree $n-1$ binomials are in the ideal generated by the quadratics of I_M so this will complete the proof.

Put $\mathbf{x}^{\mathbf{S}} = \varphi(\prod_{i=1}^{n} y_{B_i})$ and let $\mathbf{S_i}$ denote the i^{th} component of $\mathbf{S}$. We define M' to be the matroid obtained from M by replacing i with $\mathbf{S_i}$ parallel copies of i for each $i \in \{1, \ldots, m\}$; interprate 'replacing by zero parallel copies' to mean deleting this i for which $\mathbf{S_i} = 0$. There is a natural map α from the ground set of M' to the ground set of M that takes each of the parallel copies of i to i. If X is an independent set of M' then $\alpha(X)$ is an independent set of M. So there is a k-algebra homomorphism $\alpha_* : R_{M'} \longrightarrow R_M$ defined by $\alpha_*(y_{B'}) = y_{\alpha(B')}$ for every base B' of M'.

Since the collection $\mathfrak{C}$ is closed under deletions and adding parallel elements, $M \in \mathfrak{C}$ implies $M' \in \mathfrak{C}$. M' has a ground set of size $nr(M') = nr(M) = \sum_i \mathbf{S_i}$, and by assumption, the n-base graph of M' is connected. Let $\mathbf{u_B}$ be a vertex of $G_n(M')$ such that $\alpha(\mathbf{u_B}) = \{B_1, \ldots, B_n\}$ (here α is the natural extension of α to sets of subsets of the ground set on $M' : \alpha(\mathbf{u_B}) = \{\alpha(X) | X \in \mathbf{u_B}\}$). Such $\mathbf{u_B}$ exists by construction of M': simply split up the parallel copies of i, giving one to each base in $\{B_1, \ldots, B_n\}$ containing i. Let $\mathbf{u_D}$ be a vertex of $G_n(M)$ such that $\alpha(\mathbf{u_D}) = \{D_1, \ldots, D_n\}$. Let

$$\mathbf{y^u} = \prod_{X \in \mathbf{u}} y_X$$

as is customary when $\mathbf{u}$ is identified with its zero-one incidence vector. Let $\mathbf{u_0}, \mathbf{u_1}, \ldots, \mathbf{u_t}$ be the vertices of a path between $\mathbf{u_B} = \mathbf{u_0}$ and $\mathbf{u_D} = \mathbf{u_t}$ in $G_n(M')$. then, we have

$$\sum_{i=1}^{t} \mathbf{y}^{\mathbf{u_{i-1}}} - \mathbf{y}^{\mathbf{u_i}} = \mathbf{y}^{\mathbf{u_0}} - \mathbf{y}^{\mathbf{u_t}}$$

and applying the map α_* we obtain

$$\sum_{i=1}^{t} \mathbf{y}^{\alpha(\mathbf{u_{i-1}})} - \mathbf{y}^{\alpha(\mathbf{u_i})} = \mathbf{y}^{\alpha(\mathbf{u_0})} - \mathbf{y}^{\alpha(\mathbf{u_t})} = \prod_{i=1}^{n} y_{B_i} - \prod_{i=1}^{n} y_{D_i} = b. \tag{2.1}$$

For $i = 1, \ldots t$ there is a base $X \in \mathbf{u_{i_1}} \cap \mathbf{u_i}$ which implies $\alpha(X) \in \alpha(\mathbf{u_{i_1}}) \cap \alpha(\mathbf{u_i})$. This show that $y_{\alpha(X)}$ may be factored out of the binomial $\mathbf{y}^{\alpha(\mathbf{u_{i-1}})} - \mathbf{y}^{\alpha(\mathbf{u_i})}$, and therefore (2.1) shows that b is in the ideal generated by the degree $n-1$ binomials of I_M. $\square$

The reduction for Conjecture 1.4 is similar. Suppose M is a matroid on a ground set of size $2r(M)$. The *single exchange graph* of M, denoted by $G(M)$, is the graph with vertex set the set of ordered 2-tuples of bases of $M, (B_1, B_2)$ such that B_1 and B_2 are disjoint. There is an edge between (B_1, B_2) and (D_1, D_2) if and only if (D_1, D_2) can be obtained from (B_1, B_2) by a symmetric exchange (the order of the bases matters, that is, it is required $|B_1 \cap D_1| = |B_2 \cap D_2| = r(M) - 1$). The above lemma can be easily modified to show that : if for every $M \in \mathfrak{C}$ with a ground set of size $2r(M)$ the single exchange graph of M is connected, then Conjecture 1.4 holds for all matroids in $\mathfrak{C}$.

3. Strongly base orderable matroid

A matroid is *strongly base order able* if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \longrightarrow B_2$ satisfying the multiple symmetric exchange property, that is : $(B_1 \setminus A) \cup \pi(A)$ is a basis for every $A \subset B_1$. This implies that π restricted to the intersection $B_1 \cap B_2$ is the identity. Moreover, $(B_2 \setminus \pi(A)) \cup A$ is a basis for every $A \subset B_1$ (by the multiple symmetric exchange property for $B_1 \setminus A$). The class of strongly base orderable matroids is closed under taking minors.
In [11] Lasoń and Michałek proved White's conjecture for strongly base orderable matroids. As a consequence it is true for *gammoids* (every gammoid is strongly base orderable), and in particular for transversal matroids (every transversal matroid is a gammoid).

Theorem 3.1. [11] *If M is a strong orderable base matroid, then the toric ideal I_M is generated by quadratics binomials corresponding to symmetric exchanges.*

Proof. Let J_M be the ideal generated by quadratics binomials corresponding to symmetric exchanges. The ideal I_M, as a toric ideal, is generated by binomials. Thus it is enough to prove that all binomials of I_M belong to the ideal J_M.
Fix $n \geq 2$. We shall prove by decreasing induction on the overlap function

$$d(y_{B_1} \cdots y_{B_n}, y_{D_1} \cdots y_{D_n}) := \max_{\pi \in S_n} \sum_{i=1}^{n} |B_i \cap D_{\pi(i)}|$$

that a binomial $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ belongs to J_M. Clearly the biggest possible value of d is $r(M)n$ where $r(M)$ denotes the rank of M.
If $d(y_{B_1} \cdots y_{B_n}, y_{D_1} \cdots y_{D_n}) = r(M)n$ then there exists a permutation $\pi \in S_n$ such that $B_i = D_{\pi(i)}$ for each i. Hence, $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} = 0 \in J_M$.
Suppose the assertion holds for all binomials with overlap function greater that $d < r(M)n$. Let $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n}$ be a binomial of I_M (and thus $B_1 \cup \cdots \cup B_n = D_1 \cup \cdots \cup D_n$ as multisets) with the overlap function equal to d. Without loss of generality we can assume that the identity permutation realizes the maximum in the definition of the overlap function. Then for some i there exists $e \in B_i \setminus D_i$. Since $B_1 \cup \cdots \cup B_n = D_1 \cup \cdots D_n$ as multisets then there exist $j \neq i$ such that $e \in D_j \setminus B_j$. Without loss of generality we can assume that $i = 1$ and $j = 2$. Since M is strongly base orderable matroid, there exist bijections $\pi_B : B_1 \longrightarrow B_2$ and $\pi_D : D_1 \longrightarrow D_2$ with the multiple symmetric exchange property. Recall that π_B is the identity on $B_1 \cap B_2$ and similarly that π_D is the identity on $D_1 \cap D_2$.
Let G be a graph on a vertex set $B_1 \cup B_2 \cup D_1 \cup D_2$ with edges $\{b, \pi_B(b)\}$ for all $b \in B_1 \setminus B_2$ and $\{d, \pi_B(d)\}$ for all $d \in D_1 \setminus D_2$. G is bipartite since it is the sum of two matchings. Split the vertex set of G into two independent (in the graph sense) sets S

and T. Define

$$B_1' = (S \cap (B_1 \cup B_2)) \cup (B_1 \cap B_2),\ B_2' = (T \cap (B_1 \cup B_2)) \cup (B_1 \cap B_2)$$
$$D_1' = (S \cap (D_1 \cup D_2)) \cup (D_1 \cap D_2),\ D_2' = (T \cap (D_1 \cup D_2)) \cup (D_1 \cap D_2)$$

By the multiple symmetric exchange property of π_B sets B_1', B_2' are bases obtained from the pair B_1, B_2 by a sequence of symmetric exchanges. Therefore the binomial

$$y_{B_1} y_{B_2} y_{B_3} \cdots y_{B_n} - y_{B_1'} y_{B_2'} y_{B_3} \cdots y_{B_n} \tag{3.1}$$

belongs to J_M. Analogously the binomial

$$y_{D_1} y_{D_2} y_{D_3} \cdots y_{D_n} - y_{D_1'} y_{D_2'} y_{D_3} \cdots y_{D_n} \tag{3.2}$$

belongs to J_M. Moreover since S and T are disjoint we have that

$$d(y_{B_1'} y_{B_2'} y_{B_3} \cdots y_{B_n}, y_{D_1'} y_{D_2'} y_{D_3} \cdots y_{D_n}) > d(y_{B_1} y_{B_2} y_{B_3} \cdots y_{B_n}, y_{D_1} y_{D_2} y_{D_3} \cdots y_{D_n}).$$

By the inductive assumption

$$y_{B_1'} y_{B_2'} y_{B_3} \cdots y_{B_n} - y_{D_1'} y_{D_2'} y_{D_3} \cdots y_{D_n} \tag{3.3}$$

also belongs to J_M. By adding (3.1) and (3.3) and subtracting (3.2) we have that

$$y_{B_1} y_{B_2} y_{B_3} \cdots y_{B_n} - y_{D_1} y_{D_2} y_{D_3} \cdots y_{D_n}$$

belongs to J_M, as desired. □

The following three sections are based on the results given by García-Marco and Ramírez Alfonsín in [9].

4. Complete intersection

The toric ideal I_M is a *complete intersection* if $\mu(I_M) = \mathrm{ht}(I_M)$, where $\mu(I_M)$ denotes the minimal number of generators of I_M. Equivalently, I_M is a complete intersection if and only if there exists a set of homogeneous binomials $g_1, \ldots, g_s \in R$ such that $s = \mathrm{ht}(I_M)$ and $I_M = (g_1, \ldots, g_s)$.
From expression (1.1) one easily derives that whenever $r = n$ or $r = n-1$, then $I_M = (0)$ and I_M is a complete intersection. Thus, we only consider the case $r \leq n-2$.

It can be proved that the operations of taking duals, deletion, contraction and taking minors of M preserve the property of being a complete intersection on I_M. For more details on how these operations affect I_M we refer the reader to [3, Section 2].

We denote by M^* the dual matroid of M. It is straightforward to check that $\sigma(I_M) = I_{M^*}$, where σ is the isomorphism of k-algebras $\sigma : R \longrightarrow k[y_{E\setminus B} \,|\, B \in \mathcal{B}]$ induced by $y_B \mapsto y_{E\setminus B}$. Thus, I_M is a complete intersection if and only if I_{M^*} also is.

For every $A \subset E$, $M \setminus A$ denotes the *deletion of* A *from* M and M/A denotes the *contraction of* A *from* M. For $E' \subset E$, the restriction of M to E' is denoted by $M|_{E'}$.

Proposition 4.1. *Let M' be a minor of M. If I_M is a complete intersection, then $I_{M'}$ also is.*

It is not difficult to see that if e is a loop then $I_M = I_{M\setminus\{e\}}$. Moreover, if e is a coloop of M, then I_M is essentially equal to $I_{M\setminus\{e\}}$. Indeed, if one considers the isomorphism of k-algebras $\tau : R \longrightarrow k[y_{B\setminus\{e\}} \mid B \in \mathcal{B}]$ induced by $y_B \mapsto y_{B\setminus\{e\}}$, then $\tau(I_M) = I_{M\setminus\{e\}}$. For this reason we may assume without loss of generality that M has no loops or coloops.

Now we study the complete intersection property for I_M when M has rank 2. In this case, we associate to M the graph $\mathcal{H}_M$ with vertex set E and edge set $\mathcal{B}$. It turns out that I_M coincides with the toric ideal of the graph $\mathcal{H}_M$ (see, e.g., [1]). In particular, from [1, Corollary 3.9], we have that whenever I_M is a complete intersection, then $\mathcal{H}_M$ does not contain $\mathcal{K}_{2,3}$ as subgraph, where $\mathcal{K}_{2,3}$ denotes the complete bipartite graph with partitions of sizes 2 and 3. The following result characterizes the complete intersection property for toric ideals of rank 2 matroids.

Proposition 4.2. *Let M be a rank 2 matroid on a ground set of $n \geq 4$ elements without loops or coloops. Then, I_M is a complete intersection if and only if $n = 4$.*

Proof. ($\Rightarrow$) Assume that $n \geq 5$ and let us prove that I_M is not a complete intersection. Since M has no loops or coloops, we may assume that $B_1 = \{1,2\}, B_2 = \{3,4\}, B_3 = \{1,5\} \in \mathcal{B}$. Since $B_1, B_2 \in \mathcal{B}$, by the symmetric exchange axiom, we can also assume that $B_4 = \{1,3\}, B_5 = \{2,4\} \in \mathcal{B}$. If $\{4,5\} \in \mathcal{B}$, then $\mathcal{H}_M$ has a subgraph $\mathcal{K}_{2,3}$ and I_M is not a complete intersection. Let us suppose that $\{4,5\} \notin \mathcal{B}$. By the exchange axiom for B_2 and B_3 we have $B_6 := \{3,5\} \in \mathcal{B}$. Again by the exchange axiom for B_5 and B_6 we get that $B_7 := \{3,4\} \in \mathcal{B}$. Thus, $\mathcal{H}_M$ has $\mathcal{K}_{2,3}$ as a subgraph and I_M is not a complete intersection.
($\Leftarrow$) There are three non isomorphic rank 2 matroids without loops or coloops and $n = 4$. Namely, M_1 with set of bases $\mathcal{B}_1 = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$, M_2 with set of bases $\mathcal{B}_2 = \mathcal{B}_1 \cup \{\{1,4\}\}$ and $M_3 = \mathcal{U}_{2,4}$. For $i = 1,2$ one can easily check that $\mathrm{ht}(I_{M_i}) = 1$ and that $I_{M_i} = (y_{\{1,2\}}y_{\{3,4\}} - y_{\{1,3\}}y_{\{1,4\}})$; thus both I_{M_1} and I_{M_2} are complete intersections. Moreover, $\mathrm{ht}(I_{M_3}) = 2$ and a direct computation with SINGULAR or CoCoA yields that $I_{M_3} = (y_{\{1,2\}}y_{\{3,4\}} - y_{\{1,3\}}y_{\{1,4\}}, y_{\{1,4\}}y_{\{2,3\}} - y_{\{1,3\}}y_{\{1,4\}})$; thus I_{M_3} is also a complete intersection. □

One can apply Proposition 4.2 to give the list of all matroids M such that I_M is a complete intersection.

Theorem 4.3. *Let M be a matroid without loops or coloops and with $n > r+1$. Then, I_M is a complete intersection if and only if $n = 4$ and M is the matroid whose set of bases is:*

(1) $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$,
(2) $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\}\}$, *or*
(3) $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\},\{2,3\}\}$, *i.e.,* $M = \mathcal{U}_{2,4}$.

5. FINDING MINORS

In this section we investigate a characterization for a matroid to contain certain minors in terms of a set of binomial generators of its corresponding toric ideal. In particular, we focus our attention to detect if a matroid M contains $\mathcal{U}_{d,2d}$ as a minor for $d \geq 2$. We consider the following binary equivalence relation $\sim$ on the set of pairs of bases:

$$\{B_1, B_2\} \sim \{B_3, B_4\} \iff B_1 \cup B_2 = B_3 \cup B_4 \text{ as multisets,}$$

and we denote by $\Delta_{\{B_1,B_2\}}$ the cardinality of the equivalence class of $\{B_1, B_2\}$.

We now introduce two lemmas concerning the values $\Delta_{\{B_1,B_2\}}$. The first one provides some bounds on the values of $\Delta_{\{B_1,B_2\}}$. In the proof of this lemma we use the so called *multiple symmetric exchange property* (see [19]):

> For every B_1, B_2 in $\mathcal{B}$ and for every $A_1 \subset B_1$, there exists $A_2 \subset B_2$ such that $(B_1 \cup A_2) \setminus A_1 \in \mathcal{B}$ and $(B_2 \cup A_1) \setminus A_2$ are in $\mathcal{B}$.

Lemma 5.1. *For every $B_1, B_2 \in \mathcal{B}$, then $2^{d-1} \leq \Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$, where $d := |B_1 \setminus B_2|$.*

Proof. Take $e \in B_1 \setminus B_2$. By the multiple symmetric exchange property, for every A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, there exists $A_2 \subset B_2$ such that both $B_1' := (B_1 \cup A_2) \setminus A_1$ and $B_2' := (B_2 \cup A_1) \setminus A_2$ are bases. Since $B_1 \cup B_2 = B_1' \cup B_2'$ as multisets, we derive that $\Delta_{\{B_1,B_2\}}$ is greater or equal to the number of sets A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, which is exactly 2^{d-1}.
We set $A := B_1 \cap B_2$, $C := B_1 \triangle B_2$ and take $e \in B_1 \setminus B_2$. Take $B_3, B_4 \in \mathcal{B}$ such that $B_1 \cup B_2 = B_3 \cup B_4$ as multisets and assume that $e \in B_4$. Then, $B_3 \setminus A \subset C \setminus \{e\}$ with $|B_3| = |B_1 \setminus B_2| = d$ elements; thus, $\Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$. □

Moreover, the bounds of Lemma 5.1 are sharp for every $d \geq 2$. Indeed, if one considers the transversal matroid on the set $\{1, \ldots, 2d\}$ with presentation $(\{1, d+1\}, \ldots, \{d, 2d\})$, and takes the bases $B_1 = \{1, \ldots, d\}$, $B_2 = \{d + 1, \ldots, 2d\}$, then $|B_1 \setminus B_2| = d$ and $\Delta_{\{B_1,B_2\}} = 2^{d-1}$. Also, if we consider the uniform matroid $\mathcal{U}_{d,2d}$ then for any base B we have that $\Delta_{\{B,E \setminus B\}} = \binom{2d-1}{d}$.

The second lemma interprets the values of $\Delta_{\{B_1,B_2\}}$ in terms of the number of bases-cobases of a certain minor of M. Recall that a base $B \in \mathcal{B}$ is a *base-cobase* if $E \setminus B$ is also a base of M.

Lemma 5.2. *Let $B_1, B_2 \in \mathcal{B}$ of a matroid M and consider the matroid $M' := (M/(B_1 \cap B_2))|_{(B_1 \triangle B_2)}$ on the ground set $B_1 \triangle B_2$. Then, the number of bases-cobases of M' is equal to $2\Delta_{\{B_1,B_2\}}$.*

The following result provides a necessary condition for a matroid to have a minor isomorphic to $\mathcal{U}_{d,2d}$.

Proposition 5.3. *If M has a minor $M' \simeq \mathcal{U}_{d,2d}$ for some $d \geq 2$, then there exist $B_1, B_2 \in \mathcal{B}$ such that $\Delta_{\{B_1,B_2\}} = \binom{2d-1}{d}$.*

Proof. Let $A, C \subset E$ be disjoint sets such that $M' := (M \setminus A)/C \simeq \mathcal{U}_{d,2d}$ and denote $E' := E \setminus (A \cup C)$. Since $M' = (M \setminus A)/C$, then there exist $e_1, \ldots, e_{r-d} \in A \cup C$ such that $B' \cup \{e_1, \ldots, e_{r-d}\} \in \mathcal{B}$ for every B' base of M'. We take any $D \subset E'$ with d-elements, we have that $B_1 = D \cup \{e_1, \ldots, e_{r-d}\} \in \mathcal{B}$, $B_2 = (E' \setminus D) \cup \{e_1, \ldots, e_{r-d}\} \in \mathcal{B}$ and $B_1 \cup B_2 = E' \cup \{e_1, \ldots, e_{r-d}\}$. Thus, $\Delta_{\{B_1,B_2\}} \geq \binom{2d}{d}/2 = \binom{2d-1}{d}$. Since $|B_1 \setminus B_2| = d$, by Lemma 5.1 we are done. □

Since $\mathcal{U}_{2,4}$ is the only forbidden minor for a matroid to be binary the following result gives a criterion for M to be binary by proving the converse of Proposition 5.3 for $d = 2$.

Theorem 5.4. *M is binary if and only if $\Delta_{\{B_1,B_2\}} \neq 3$ for every $B_1, B_2 \in \mathcal{B}$.*

Proof. ($\Rightarrow$) Assume that there exist $B_1, B_2 \in \mathcal{B}$ such that $\Delta_{\{B_1,B_2\}} = 3$. Let us denote $d := |B_1 \setminus B_2|$. By Lemma 5.1 we observe that $d = 2$. If we set $C := B_1 \cap B_2$ and $A = E \setminus (B_1 \cup B_2)$, then $M' := (M \setminus A)/C$ is a rank 2 matroid on a ground set of 4 elements and, by Lemma 5.2, it has 6 bases-cobases, thus $M' \simeq \mathcal{U}_{2,4}$ and M is not binary.
($\Leftarrow$) Assume that M is not binary, then M has a minor $M' \simeq \mathcal{U}_{2,4}$ and the result follows from Proposition 5.3. □

It can also be proved that the converse of Proposition 5.3 also holds for $d = 3$.

Theorem 5.5. *M has a minor $M' \simeq \mathcal{U}_{3,6}$ if and only if $\Delta_{\{B_1,B_2\}} = 10$ for some $B_1, B_2 \in \mathcal{B}$.*

6. Minimal system of generators

Minimal systems of binomial generators of toric ideals have been studied in several papers; see, e.g., [4, 8]. In general, for a toric ideal it is possible to have more than one minimal system of generators formed by binomials. Given a toric ideal I, we denote by $\nu(I)$ the number of minimal sets of binomial generators of I, where the sign of a binomial does not count and we denote by $\mu(I_M)$ the minimal number of generators of I_M.
We first give some bounds for the values of $\mu(I_M)$ and $\nu(I_M)$ in terms of the values $\Delta_{\{B_1,B_2\}}$ for $B_1, B_2 \in \mathcal{B}$. Moreover, this lower bounds turn out to be the exact values if I_M is generated by quadrics.

Theorem 6.1. *Let $R = \{\{B_1, B_2\}, \ldots, \{B_{2s-1}, B_{2s}\}\}$ be a set of representatives of $\sim$ and set $r_i := \Delta_{\{B_{2i-1},B_{2i}\}}$ for all $i \in \{1, \ldots, s\}$. Then,*

1. *$\mu(I_M) \geq (b^2 - b - 2s)/2$, where $b := |\mathcal{B}|$, and*
2. *$\nu(I_M) \geq \prod_{i=1}^{s} r_i^{r_i-2}$.*

Moreover, in both cases equality holds whenever I_M is generated by quadrics.

We end by characterizing all matroids whose toric ideal has a unique minimal binomial generating set. We recall that the *basis graph of a matroid M* is the undirected graph $\mathcal{G}_M$ with vertex set $\mathcal{B}$ and edges $\{B, B'\}$ such that $|B \setminus B'| = 1$. We also recall that the *diameter of a graph* is the maximum distance between two vertices of the graph.

Theorem 6.2. *Let M be a rank $r \geq 2$ matroid. Then, $\nu(I_M) = 1$ if and only if M is binary and the diameter of $\mathcal{G}_M$ is at most 2.*

Proof. ($\Rightarrow$) By Theorem 6.1,we have that $\Delta_{\{B_1,B_2\}} \in \{1, 2\}$ for all $B_1, B_2 \in \mathcal{B}$. By Lemma 5.1 and Theorem 5.4, this is equivalent to M is binary and $|B_1 \setminus B_2| \in \{1, 2\}$ for all $B_1, B_2 \in \mathcal{B}$. Clearly this implies that the diameter of $\mathcal{G}_M$ is less or equal to 2.
($\Leftarrow$) Assume that the diameter of $\mathcal{G}_M$ is ≤ 2, we claim that M is strongly base orderable. Recall that a matroid is strongly base orderable if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \to B_2$ such that $(B_1 \setminus A) \cup \pi(A)$ is a basis for all $A \subset B_1$. We take $B_1, B_2 \in \mathcal{B}$ and observe that $|B_1 \setminus B_2| \in \{1, 2\}$. If $B_1 \setminus B_2 = \{e\}$ and $B_2 \setminus B_1 = \{f\}$ if suffices to consider the bijection $\pi : B_1 \to B_2$ which is the identity on $B_1 \cap B_2$ and $\pi(e) = f$. Moreover, if $B_1 \setminus B_2 = \{e_1, e_2\}$ and $B_2 \setminus B_1 = \{f_1, f_2\}$, we denote $A := B_1 \cap B_2$ and, by the symmetric exchange axiom, we can assume that both $A \cup \{e_1, f_1\}$ and $A \cup \{e_2, f_2\}$ are basis of M; then it suffices to consider $\pi : B_1 \to B_2$ the identity on A, $\pi(e_1) = f_2$ and $\pi(e_2) = f_1$ to conclude that M is strongly base orderable.

So, by [11, Theorem 2], I_M is generated by quadrics. Moreover, from Lemma 5.1 and Theorem 5.4 we deduce that $\Delta_{\{B_1,B_2\}} \in \{1,2\}$ for all $B_1, B_2 \in \mathcal{B}$. Hence, the result follows by Theorem 6.1. $\square$

References

[1] I. Bermejo, I. García-Marco, E. Reyes, Graphs and complete intersection toric ideals, J. Algebra Appl. (2014), to appear.

[2] J. Blasiak, The toric ideal of a graphic matroid is generated by quadratics, Combinatorica 28 (2008), 283–297.

[3] S. Blum, Base-sortable matroids and Koszulness of semigroup rings, European J. Combin. 22(7) (2001), 937–951.

[4] E. Briales, A. Campillo, C. Marijuán, P. Pisón, Minimal systems of generators for ideals of semigroups. J. Pure Appl. Algebra 124 (1998), no. 1-3, 7–30.

[5] R. A. Brualdi, Comments on bases in dependence structures, Bull. Austral. Math. Soc. 1 (1969), 161–167.

[6] J. Bonin, Basis-exchange properties of sparse paving matroids, Adv. in Appl. Math. 50 (2013)(1), 6–15.

[7] A. Conca, Linear spaces, transversal polymatroids and ASL domains, J. Algebraic Comb. 25 (2007), 25–41.

[8] P. Diaconis, B. Sturmfels, Algebraic algorithms for sampling from conditional distributions, Ann. Statist. 26 (1998), no. 1, 363–397.

[9] I. García-Marco and J.L. Ramírez Alfonsín, Matroid toric ideal: complete intersection, minors and minimal systems of generators, arXiv:1408.0419.

[10] K. Kashiwabara, The toric ideal of a matroid of rank 3 is generated by quadratics, Electron. J. Comb. 17 (2010), RP28, 12 pp.

[11] M. Lasoń, M. Michałek, On the toric ideal of a matroid, Adv. Math. 259 (2014), 1–12.

[12] Y. Matsumoto, S. Moriyama, H. Imai, D. Bremner, Matroid enumeration for incidence geometry, Discrete and Computational Geometry, 47 (2012), issue 1, pp. 17–43.

[13] H. Ohsugi, J. Herzog, T. Hibi, Combinatorial pure subrings, Osaka J. Math. 37 (2000), 745–757.

[14] J. Schweig, Toric ideals of lattice path matroids and polymatroids, J. Pure Appl. Algebra 215 (2011), 2660-2665.

[15] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, University Lecture Series 8, American Mathematical Society, Providence, RI, 1996.

[16] B. Sturmfels, Equations defining toric varieties, Proc. Sympos. Pure Math. 62 (1997), 437–449.

[17] N. White, The bases monomial rings of a matroid. Adv. Math. 24 (1977), 292–297.

[18] N. White, A Unique Exchange Property for Bases. Linear Algebra Appl. 31 (1980), 81–91.

[19] D. R. Woodall, An exchange theorem for bases of matroids, J. Combin. Theory Ser. B 16 (1974), 227–228.

I3M, Université Montpellier 2, Place Eugène Bataillon, 34095 Montpellier, France
E-mail address: jramirez@univ-montp2.fr