LECTURE ON 0-SEQUENCES AND h-VECTORS OF MATROID SIMPLICIAL COMPLEXES

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Abstract. In this notes, we give a basic introduction to matroids simplicial complexes.

1. Introduction

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a *simplicial complex* if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

Elements of the simplicial complex Δ are called *faces* of Δ. Maximal faces (under inclusion) are called *facets*. If $F \in \Delta$ then the *dimension* of F is $\dim F = |F| - 1$. The dimension of Δ is defined to be $\dim \Delta = \max\{\dim F | F \in \Delta\}$. The complex Δ is said to be *pure* if all its facets have the same dimension.

If $\{v\} \in \Delta$ then we call v a *vertex* of Δ. Throughout this notes Δ will denote a simplicial complex with $\{1, \ldots, n\}$.

Let $d - 1 = \dim\Delta$. The *f-vector* of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta| \dim F = i\}|$ is the number of i-dimensional faces in Δ.

Let Δ be a simplicial complex with vertex set X.

(a) The k-skeleton of Δ is $[\Delta_k] = \{F \in \Delta | \dim F \leq k\}$.

(b) If $W \subseteq X$ then the *restriction* of Δ to W is $\Delta|_W = \{F \in \Delta | F \subseteq W\}$. If $W = X - \{v\}$ then we will write $\Delta_{-v} = \Delta|_W$ and call Δ_{-v} the *deletion* of Δ with respect to v or the deletion of v from Δ.

(c) If $F \subseteq X$ then $link_\Delta(F) = \{G \in \Delta | F \cap G = \emptyset, F \cup G \in \Delta\}$. We call this the *link* of Δ with respect to F.

(d) If $v \notin X$ then the *cone* over Δ is $C\Delta = \Delta \cup \{F \cup \{v\} | F \in \Delta\}$

That all of these are again simplicial complexes is easily checked using the definition. Since if $G \in \Delta$ and $F \subseteq G$ then $F \in \Delta$, the complex Δ is determined completely by those faces that are not contained in any other face, that is the facets of Δ. Typically, we will describe a simplicial complex by listing its facets.

Example 1.1. Figure 1 illustrates a simplicial complexe Δ of dimension 2. Δ is not pure as it has facets with dimension 0 (1, 2, 3, 4 and 5), of dimension 1 (12, 13,15, 23, 24, 34, 35 and 45) and of dimension 2 (234 and 135). $f(\Delta) = (5, 8, 2)$. The link of Δ with respect to the vertex 3 is the complex with 15 and 24, while the link with respect to the vertex 5 has facets 13 and 4. The deletion of 3 has facets 12, 24, 45 and 15. The deletion of 5 has facets 234, 13 and 12.

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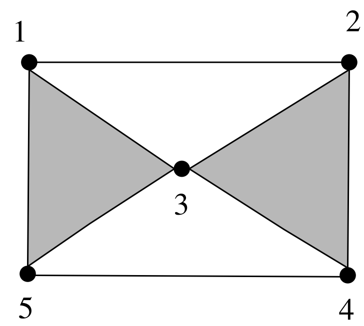

FIGURE 1. Simplicial complexe Δ

Let k be a field. We can associate to a simplicial complex Δ, a square free monomial ideal in $S = k[x_1, \ldots, x_n]$,

$$I_\Delta = \left(x_F = \prod_{i \in F} x_i | F \notin \Delta \right) \subseteq S.$$

The ideal I_Δ is called the *Stanley-Reisner ideal* of Δ.

2. Matroid simplicial complexes

Recall that axioms $(I1), (I2)$ for the independent set $\mathcal{I}(M)$ of a matroid M on V are equivalent to $\mathcal{I}$ being an abstract simplicial complex on V. The independent sets of M form a simplicial complexe, called the *independence complex* of M. One can replace the exchange axiom $(I3)$ with various others. For example, it can be replaced by the following axiom.
$(I3)'$ for every $A \subset E$ the restriction

$$\mathcal{I}|_A = \{I \in \mathcal{I} : I \subset A\}$$

is a *pure* simplicial complex.

A simplicial complexe Δ over the vertices V is called *matroid complex* if axiom $(I3)'$ is verified. A facet of a Δ are the bases of the matroids, and a minimal non-face of the complexe (or *missing face*) is a circuit, which correspond bijectively to a minimal monomial generator of the ideal I_Δ. The rank of of the matroid is equal to $\dim \Delta + 1$. There are several equivalent definitions of a matroid complexe, for instance the one given by circuit exchange property : Δ is a matroid complex if and only if for any two minimal generators M and N of I_Δ their least common multiple divided by any variable in the support of both M and N is in I_Δ (see [9]).

Example 2.1. The complex depicted in Figure 2 (a) is matroid (this can be checked by verifying that every subset [6] is pure) while the one in Figure 2 (b) is not since it admits a restriction that is not pour, for instance, the facets of $\Delta_{1,3,4}$ are $\{1\}$ and $\{3, 4\}$ as facets so the restriction is not pure.

We summarize some of the more standard constructions for a matroid complex in the next proposition.

Proposition 2.2. *Let Δ be a matroid complex with vertex set $[n]$. Then the following complexes are also matroid.*

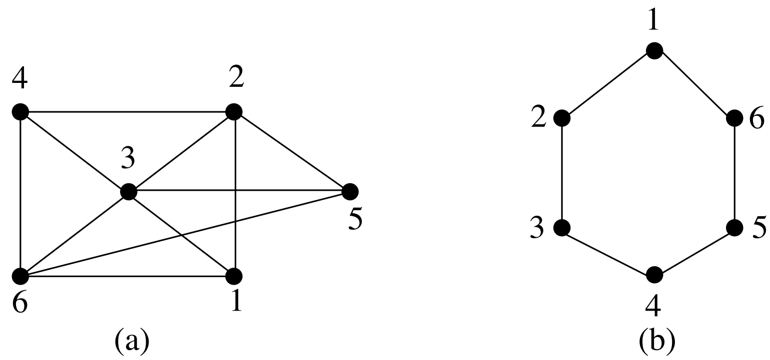

FIGURE 2. (a) Matroid complexe with 6 vertices and (b) A non-matroid complex with 6 vertices.

(a) $\Delta|_W$ *for every* $W \subseteq [n]$.
(b) $C\Delta$*, the cone over* Δ.
(c) $[\Delta]_k$*, the* k*-skeleton of* Δ.
(d) $link_\Delta(F)$ *for every* $F \in \Delta$.

Proof. (a) Since $(\Delta|_W)|_V = \Delta|_{W\cap V}$ and the left-hand side is, by definition, pure this follows immediately from the definition.

(b) Let v be the vertex of the cone. Clearly the cone over a pure complex is pure, so let $W \subset [n]$. If $v \notin W$ then $(C\Delta)|_W = \Delta|_W$, which is pure because Δ is matroid. If $v \in W$ then $(C\Delta)|_W = C(\Delta|_W)$. By part (a) $\Delta|_W$ is matroid and so, by induction on the number of vertices, $C(\Delta|_W)$ is matroid and in particular pure.

(c) Note that $[\Delta|_W]_k = [\Delta]_k|_W$. As in part (b), if W is a proper subset of $[n]$ then this is matroid, and thus pure, by induction on the number of vertices. It only remains to check that $[\Delta]_k$ is itself pure. Suppose that $[\Delta]_k$ has a face F with dim $F < k$. Since $F \in \Delta$ it must be contained in some facet with dimension dim $\Delta \geq k$. It then follows that F must be contained in some k-dimensional face of Δ, which is then a face of $[\Delta]_k$. Thus F is not a facet of $[\Delta]_k$ and the k-skeleton is therefor pure.

(d) This time, we check that $link_\Delta(F)|_W = link_{\Delta|_W}(F)$, which will then be pure by induction. We then only need to know that $link_\Delta(F)$ is pure. Suppose that $G \in link_\Delta(F)$ is a facet. Then, $G \cup F \in \Delta$ must be a facet of Δ. So dim $(G \cup F)$ =dimΔ and then dim G =dim $\Delta-$ dim$F - 1$ =dim $link_\Delta(F)$. So the link is pure and thus matroid. □

Link and deletion are identical to the contraction and deletion constructions from matroids. Notice that a complex Γ is a cone with apex x if and only if x is contained in all the facets of Γ. A matroid is a cone if and only if it has a coloop, which corresponds to the apex defined above.

Lemma 2.3. *Let* Δ *be a 1-dimensional simplicial complex. Then* Δ *is matroid if and only if for every vertex* v *and every edge* E*,* $link_\Delta(v) \cap E \neq \emptyset$.

Proof. Suppose there exists a vertex v and an edge E disjoint from the link of v. Let $L = [n] - link_\Delta(v)$. Then Δ_L has $\{v\}$ and E as facets, and so is not matroid.
Conversely, suppose that there exists a subset $W \subseteq [n]$ such that Δ_W is not pure. So Δ_W must have a 0-dimensional facet, say $\{v\}$. Let $v \neq w \in W$. Since v is a facet of

Δ_W we must have $\{v, w\} \notin \Delta$. Thus $W \cap link_W(v) = \emptyset$ and so any edge, E, of Δ_W (there must be at least one since Δ_W is not pure) must also be disjoint from $link_\Delta(v)$. Since E is also an edge of Δ the proof is complete. □

3. h-vector of simplicial complexes

Assume that dim $\Delta = d - 1$. We study the *h-vector of a simplicial complexe* of Δ $h(\Delta) = (h_0, \dots, h_d)$ from its f-vector via the relation

$$\sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} = \sum_{i=0}^{d} h_i t^i \tag{3.1}$$

In particular, for any $j = 0, \dots, d$, we have

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-1} h_i$$

and

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-1} f_{i-1}.$$

It should not be expected that the h-numbers of an arbitrary simplicial complex are nonnegative; however, the h-number of a matroid M may be interpreted combinatorially in terms of certain invariants of M. Fix a total ordering $\{v_1, < v_2 < \cdots < v_n\}$ on $E(M)$. Given a bases B, an element $v_j \in B$ is *internally passive* in B if there is some $v_i \in E \setminus B$ such that $v_i < v_j$ and $(B \setminus v_j) \cup v_i$ is a bases of M. Dually, $v_j \in E \setminus B$ is *externally passive* in B if there is some $v_i \in B$ such that $v_i < v_j$ and $(B \setminus v_i) \cup v_j$ is a bases of M. Alternatively, v_j is externally passive in B if it is internally passive in $E \setminus B$ in M^*. It is well known that [1]

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M)} t^{ip(B)} \tag{3.2}$$

where $ip(B)$ counts the number of internally passive elements in B. This proves that the h-numbers of a matroid complex are nonnegative. Alternatively,

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M^*)} t^{ep(B)} \tag{3.3}$$

where $ep(B)$ counts the number of externally passive elements in B. Since the f-numbers (and hence the h-numbers) of a matroid depend only on its independent sets, equations (3.2) and (3.3) hold for *any* ordering of the ground set of M. We also remark that the h-vector of a matroid complex Δ_M is actually a specialization of the Tutte polynomial of the corresponding matroid; precisely we have $T(M; x, 1) = h_0 x^d + h_1 x^{d_1} + \cdots + h_d$, see [1].

Example 3.1. We consider the matroid complexe $\Delta(U_{2,3})$ associated to the uniform matroid of rank 2 with 3 elements. We shall calculate the h-vector of Δ via relations (3.1) , (3.2), (3.3) and also via the Tutte polynomial associated to $U_{2,3}$.
We have that dim $\Delta = 1$ and $f_{-1} = 1, f_0 = 3$ and $f_1 = 3$. Therefore by relation (3.1), we have

$$\begin{aligned}\sum_{i=0}^{2} f_{i-1}t^i(1-t)^{2-i} &= f_{-1}t^0(1-t)^2 + f_0t(1-t) + f_1t^2(1-t)^0\\ &= (1-t)^2 + 3t(1-t) + 3t^2\\ &= 1-2t+t^2+3t-3t-3t^2+3t^2 = t^2+t+1 = \sum_{i=0}^{2} h_it^i.\end{aligned}$$

Thus $h = (1, 1, 1)$.

Let $\mathcal{B}(U_{2,3}) = \{B_1 = \{1, 2\}, B_2 = \{1, 3\}, B_3 = \{2, 3\}\}$. We can check that
- there is not internally passive element in B_1
- 3 is internally passive element of B_2
- 2 and 3 are internally passive elements of B_3
Thus by (3.2) we have

$$\sum_{i=0}^{2} h_it^i = \sum_{B\in\mathcal{B}(U_{2,3})} t^{ip(B)} = 1 + t + t^2.$$

Let $\mathcal{B}(U^*_{2,3} = U_{1,3}) = \{B_1 = \{1\}, B_2 = \{2\}, B_3 = \{3\}\}$. We can check that
- 2 and 3 are externally passive elements of B_1
- 3 is externally passive element of B_2
- there is not externally passive element in B_3
Thus by (3.3) we have

$$\sum_{i=0}^{2} h_it^i = \sum_{B\in\mathcal{B}(U_{1,3})} t^{ep(B)} = t^2 + t + 1.$$

Let us now calculate $T(U_{3,2}; x, y)$. We recursively have

$$\begin{aligned}T(U_{3,2}; x, y) &= T(U_{3,2} \setminus 3; x, y) + T(U_{3,2}/3; x, y)\\ &= T(U_{2,2}; x, y) + T(U_{2,1}; x, y).\end{aligned}$$

Since $U_{k,k}$, $k \geq 1$ consiste of one base so all its elements are isthmes, so

$$\begin{aligned}T(U_{2,2}; x, y) &= T(U_{2,2}(2); x, y)T(U_{2,2} \setminus 2; x, y)\\ &= T(I; x, y)T(U_{1,1}; x, y)\\ &= xT(U_{1,1}; x, y)\\ &= xT(I; x, y) = x^2\end{aligned}$$

and

$$\begin{aligned}T(U_{2,1}; x, y) &= T(U_{2,1} \setminus 2; x, y) + T(U_{2,1}/2; x, y)\\ &= T(U_{1,1}; x, y) + T(U_{1,0}; x, y)\\ &= T(I; x, y) + T(B; x, y)\\ &= x + y.\end{aligned}$$

Obtaining

$$T(U_{3,2}; x, y) = x^2 + x + y,$$

and thus

$$T(U_{3,2}; t, 1) = t^2 + t + 1 = \sum_{i=0}^{2} h_it^i.$$

4. Stanley's conjecture

An *order ideal* $\mathcal{O}$ is a family of monomials (say of degree at most r) with the property that if $\mu \in \mathcal{O}$ and $\nu|\mu$ then $\nu \in \mathcal{O}$. Let $\mathcal{O}_i$ denote the collection of monomials in $\mathcal{O}$ of degree i. Let $F_i(\mathcal{O}) := |\mathcal{O}_i|$ and $F(\mathcal{O}) = (F_0(\mathcal{O}), F_1(\mathcal{O}), \dots, F_r(\mathcal{O}))$. We say that $\mathcal{O}$ is *pure* if all its maximal monomials (under divisibility) have the same degree. A vector $\mathbf{h} = (h_0, \dots, h_d)$ is a *pure O-sequence* if there is a pure ideal $\mathcal{O}$ such that $\mathbf{h} = F(\mathcal{O})$.

Example 4.1. The pure monomial order ideal (inside $k[x, y, z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

$$\begin{aligned} X = \ & \{\mathbf{xy^3z}, \mathbf{x^2z^3}; y^3z, xy^2z, xy^3, xz^3, x^2z^2, y^2z, y^3, xyz, \\ & xy^2, xz^2, z^3, x^2z, yz, y^2, xz, xy, z^2, x^2, z, y, x, 1\}. \end{aligned}$$

Hence the h-vector of X is the pure O-sequence $h = (1, 3, 6, 7, 5, 2)$.

A longstanding conjecture of Stanley [10] suggest that matroid h-vectors are highly structured

Conjecture 4.2. *For any matroid M, $h(M)$ is a pure O-sequence.*

Conjecture 4.2 is known to hold for several families of matroid complexes such as paving matroids [6], cographic matroids [5], cotraversal matroids [7], lattice path matroids [8], matroids of rank at most three [2, 3, 11] and for all matroids on at most nine elements all matroids of corank two [2].

4.1. **Rank-2 Case.** Let M be a loopless matroid of rank 2. The independence complex of M is a complete graph whose partite sets $E_1, \dots, E_t$ are the parallelism classes of M. let $s_i := |E_i|$. Choose one representative $e_i \in E_i$ from each parallelism calsses of M so that the simplification of M is a complete graph of $\{e_1, \dots, e_t\}$, and let $\tilde{E}_i = E_i \setminus e_i$. Clearly,

$$f_0(M) = \sum_{i=1}^{t}(s_i - 1) + t$$

and

$$f_1(M) = \sum_{1 \le i < j \le t} (s_i - 1) + (s_j - 1) + (t - 1)\sum_{i=1}^{t}(s_i - 1) + \binom{t}{2},$$

and hence,

$$h_1(M) = \sum_{i=1}^{t}(s_i - 1) + (t - 2)$$

and

$$h_2(M) = \sum_{1 \le i < j \le t} (s_i - 1) + (s_j - 1) + (t - 1)\sum_{i=1}^{t}(s_i - 1) + \binom{t-1}{2}.$$

Consider the pure O-sequence $\mathcal{O}$ with

$$\begin{aligned} \mathcal{O}_1 \ & = \{x_1, \dots, x_{t-2}\} \cup \{x_e : e \in \tilde{E}_i, 1 \le i \le t\} \\ \mathcal{O}_2 \ & = \{x_e x_{e'} : e \in \tilde{E}_i, e' \in \tilde{E}_j, 1 \le i < j \le t\} \\ & \cup \{x_i x_e : e \in \tilde{E}_j, 1 \le i \le t-2, 1|ej \le t\} \\ & \cup \{\text{ degree 2 monomials in } x_1, \dots x_{t-2}\}. \end{aligned}$$

We see that $h(M) = f(\mathcal{O})$, which proves that $h(M)$ is a pure O-sequence.

Example 4.3. We consider the matroid complexe Δ associated to the rank 2 matroid induced by the graph G illustrated in figure 3.

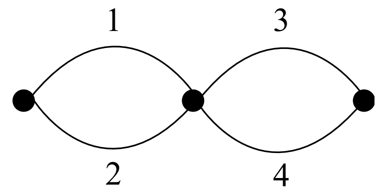

FIGURE 3. Graph G.

We have that dim $\Delta = 1$ and $f_{-1} = 1, f_0 = 4$ and $f_1 = 4$. Let $\mathcal{B}(M(G)) = \{B_1 = \{1,3\}, B_2 = \{1,4\}, B_3 = \{2,3\}, B_4 = \{2,4\}\}$. We can check that
- there is not internally passive element in B_1
- 4 is internally passive element of B_2
- 2 is internally passive element of B_3
- 2 and 4 are internally passive elements of B_4
Thus by (3.2) we have

$$\sum_{i=0}^{2} h_i t^i = \sum_{B \in \mathcal{B}(M(G))} t^{ip(B)} = 1 + t + t + t^2 = 1 + 2t + t^2.$$

Obtaining the h-vector $h(1,2,1)$. Since $\mathcal{O} = (1, x_1, x_2, x_1x_2)$ is an order ideal then $h(1,2,1)$ is pure O-sequence.

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