Toric ideals and matroids I

J.L. Ramírez Alfonsín

I3M, Université Montpellier 2

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A binomial in R is a difference of two monomials, an ideal generated by binomials is called a *binomial ideal*.

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We consider the homomorphism of k-algebras $\varphi : R \longrightarrow k[x_1, \ldots, x_n]$ induced by

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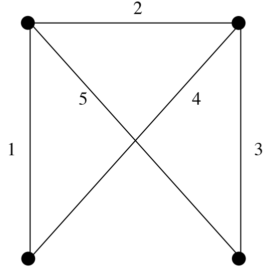
Observation Let b be the number of bases of a matroid M on n elements. Then, I_M is generated by the kernel of the integer $n \times b$ matrix whose columns are the zero-one incidence vectors of the bases of M.

Example

Matroid $M(G)$ associated to graph G. We have $r(M(G)) = 3$.



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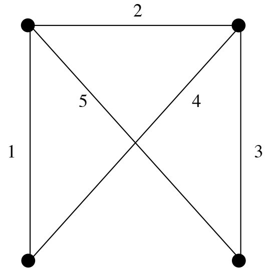

$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$

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By considering $\varphi : k[y_{B_1}, \ldots, y_{B_8}] \longrightarrow k[x_1, \ldots, x_5]$ we have that

$y_{B_1} \mapsto x_1x_2x_3, \quad y_{B_2} \mapsto x_1x_2x_5, \quad y_{B_3} \mapsto x_1x_3x_4, \quad \ldots$

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An element of the kernel of φ ($i.e.$, $I_{M(G)}$) is : $y_{B_7}y_{B_4} - y_{B_2}y_{B_8}$.

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Observation Since $R/I_M \simeq S_M$, it follows that the height of I_M is $\mathrm{ht}(I_M) = |\mathcal{B}| - \dim(S_M) = |\mathcal{B}| - (n - c + 1)$, where c is the number of connected components of M.

White's conjecture

Let $\mathcal{B}$ denote the set of bases of M. By definition $\mathcal{B}$ is not empty and satisfies the following exchange axiom :

> *For every $B_1, B_2 \in \mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$.*

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Brualdi proved that the exchange axiom is equivalent to the symmetric exchange axiom :

For every B_1, B_2 in $\mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that both $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$ and $(B_2 \cup \{e\}) \setminus \{f\} \in \mathcal{B}$.

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Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$.

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Conjecture (White 1980) For every matroid M its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.

White's conjecture

Observation for $B_1, \ldots, B_s, D_1, \ldots, D_s \in \mathcal{B}$, the homogeneous binomial $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$ belongs to I_M if and only if $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$ as multisets.

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Since I_M is a homogeneous binomial ideal, it follows that

$$I_M=\left(\{y_{B_1}\cdots y_{B_s}-y_{D_1}\cdots y_{D_s}\mid B_1\cup\cdots\cup B_s=D_1\cup\cdots\cup D_s \text{ as multisets}\}\right)$$

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Observation White's conjecture does not depend on the field k.

Example continued

We had $\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$.

We also had that $y_{B_7}y_{B_4} - y_{B_2}y_{B_8} \in I_{M(G)}$.

We can check that $B_7 \cup B_4 = \{2, 4, 5, 1, 3, 5\} = B_2 \cup B_8$.

Results of White's conjecture

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- Lasoń, Michałek (2014) proved for strongly base orderables matroids.

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The *n*-base graph of M, which is denoted by $G_n(M)$, has as its vertex set the set of all sets of n disjoint bases (a set of n bases $\{B_1, \ldots, B_n\}$ of M is disjoint if and only if

$$|E| = \bigcup_{i=1}^{n} B_i.$$

There is an edge between $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ if and only if $B_i = D_j$ for some i, j.

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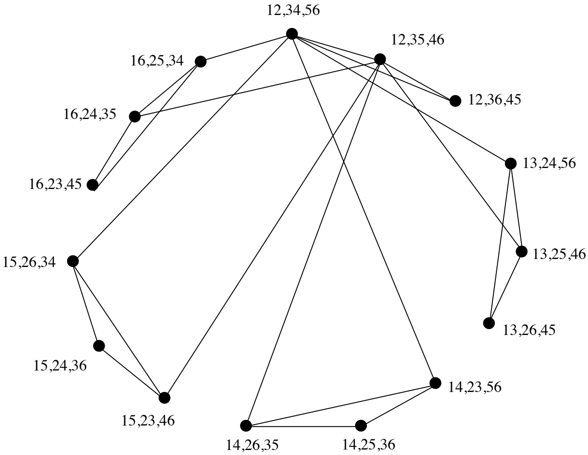

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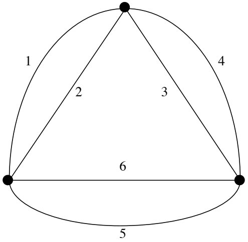

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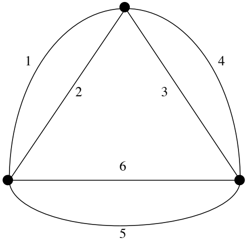

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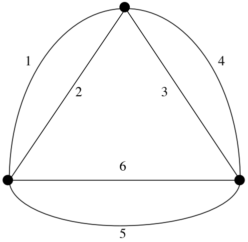

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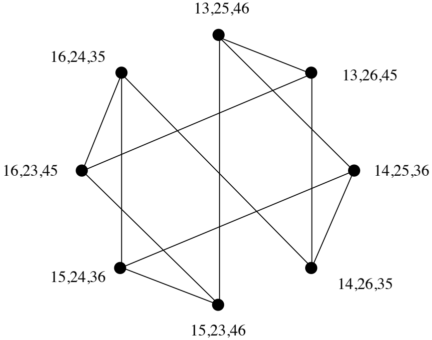

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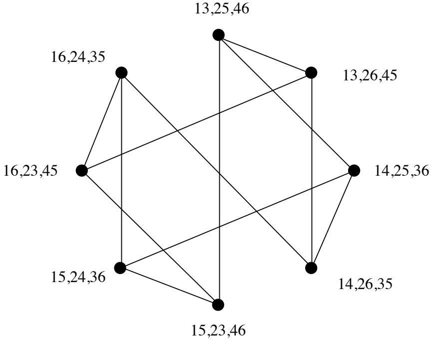

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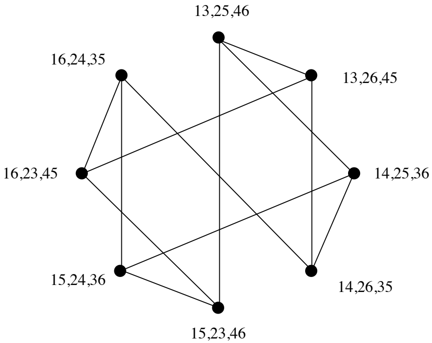

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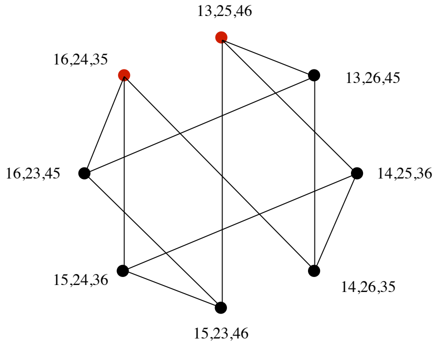

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Blasiak's reduction

Lemma (Blasiak) Let $\mathfrak{C}$ be a collection of matroids that is closed under deletions and adding parallel elements. Suppose that for each $n \geq 3$ and for every matroid M in $\mathfrak{C}$ on a ground set of size $nr(M)$ the n-base graph of M is connected. Then, for every matroid M in $\mathfrak{C}$, I_M is generated by quadratics polynomials.

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This will prove the result because I_M, as a toric ideal, is generated by binomials.

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The binomial b is necessarily of the form $b = \prod_{i=1}^{n} y_{B_i} - \prod_{i=1}^{n} y_{D_i}$ for some bases $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ of M such that the B_i and D_i have the same multiset union.

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By induction the degree $n-1$ binomials are in the ideal generated by the quadratics of I_M so this will complete the proof.

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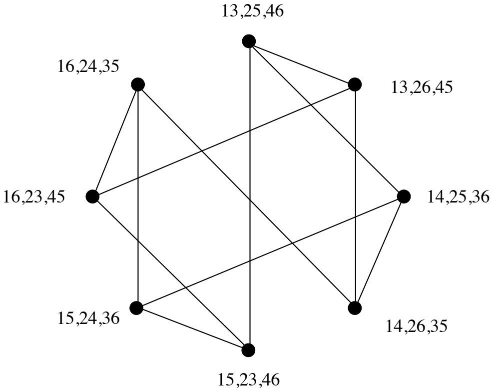

$y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

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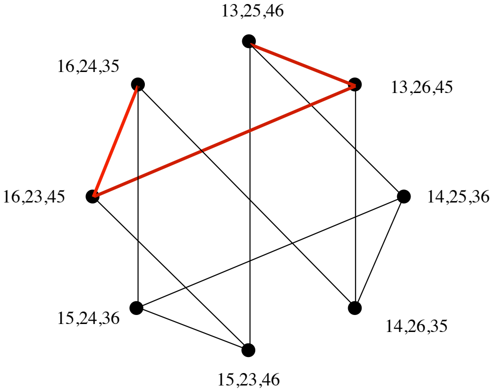

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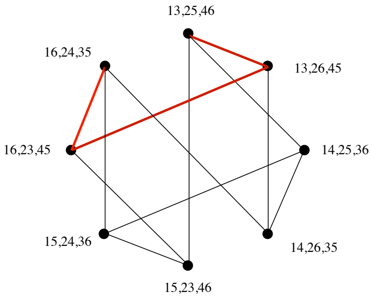

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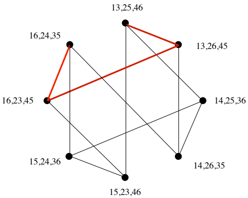

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Or equivalently
$y_{16}(y_{24}y_{35} - y_{23}y_{45}) + y_{45}(y_{16}y_{23} - y_{13}y_{26}) + y_{13}(y_{26}y_{55} - y_{25}y_{46}) = y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

Strongly base orderable matroid

A matroid is strongly base order able if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \longrightarrow B_2$ satisfying the multiple symmetric exchange property, that is : $(B_1 \setminus A) \cup \pi(A)$ is a basis for every $A \subset B_1$.

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- The class of strongly base orderable matroids is closed under taking minors.

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Fix $n \geq 2$. We shall prove by decreasing induction on the overlap function

$$d(y_{B_1} \cdots y_{B_n}, y_{D_1} \cdots y_{D_n}) := \max_{\pi \in S_n} \sum_{i=1}^{n} |B_i \cap D_{\pi(i)}|$$

that a binomial $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ belongs to J_M.

Strongly base orderable matroid

Proof (Cont...) If $d(y_{B_1}\cdots y_{B_n}, y_{D_1}\cdots y_{D_n}) = r(M)n$ then there exists a permutation $\pi \in S_n$ such that $B_i = D_{\pi(i)}$ for each i. Hence, $y_{B_1}\cdots y_{B_n} - y_{D_1}\cdots y_{D_n} = 0 \in J_M$.

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Hence, $y_{B_1}\cdots y_{B_n} - y_{D_1}\cdots y_{D_n} = 0 \in J_M$.

Suppose the assertion holds for all binomials with overlap function greater that $d < r(M)n$. Let $y_{B_1}\cdots y_{B_n} - y_{D_1}\cdots y_{D_n}$ be a binomial of I_M with the overlap function equal to d.

Strongly base orderable matroid

Proof (Cont...) Since M is strongly base orderable matroid, there exist bijections $\pi_B : B_1 \longrightarrow B_2$ and $\pi_D : D_1 \longrightarrow D_2$ with the multiple symmetric exchange property.

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Let G be a graph on a vertex set $B_1 \cup B_2 \cup D_1 \cup D_2$ with edges $\{b, \pi_B(b)\}$ for all $b \in B_1 \setminus B_2$ and $\{d, \pi_B(d)\}$ for all $d \in D_1 \setminus D_2$.

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G is bipartite since it is the sum of two matchings. Split the vertex set of G into two independent (in the graph sense) sets S and T.

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We define

$$B_1' = (S \cap (B_1 \cup B_2)) \cup (B_1 \cap B_2), \; B_2' = (T \cap (B_1 \cup B_2)) \cup (B_1 \cap B_2)$$
$$D_1' = (S \cap (D_1 \cup D_2)) \cup (D_1 \cap D_2), \; D_2' = (T \cap (D_1 \cup D_2)) \cup (D_1 \cap D_2)$$

Strongly base orderable matroid

Proof (Cont...) By the multiple symmetric exchange property of π_B sets B'_1, B'_2 are bases obtained from the pair B_1, B_2 by a sequence of symmetric exchanges. Therefore the binomial

$$y_{B_1} y_{B_2} y_{B_3} \cdots y_{B_n} - y_{B'_1} y_{B'_2} y_{B_3} \cdots y_{B_n} \tag{1}$$

belongs to J_M.

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belongs to J_M.

Analogously, the binomial

$$y_{D_1}y_{D_2}y_{D_3}\cdots y_{D_n} - y_{D'_1}y_{D'_2}y_{D_3}\cdots y_{D_n} \quad (2)$$

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Proof (Cont...) By the multiple symmetric exchange property of π_B sets B'_1, B'_2 are bases obtained from the pair B_1, B_2 by a sequence of symmetric exchanges. Therefore the binomial

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belongs to J_M.

Moreover since S and T are disjoint we have that

$$d(y_{B'_1} y_{B'_2} y_{B_3} \cdots y_{B_n}, y_{D'_1} y_{D'_2} y_{D_3} \cdots y_{D_n}) > d(y_{B_1} y_{B_2} y_{B_3} \cdots y_{B_n}, y_{D_1} y_{D_2} y_{D_3} \cdots y_{D_n})$$

Strongly base orderable matroid

Proof (Cont...) By the inductive assumption

$$y_{B'_1} y_{B'_2} y_{B_3} \cdots y_{B_n} - y_{D'_1} y_{D'_2} y_{D_3} \cdots y_{D_n} \qquad (3)$$

also belongs to J_M.

Strongly base orderable matroid

Proof (Cont...) By the inductive assumption

$$y_{B_1'} y_{B_2'} y_{B_3} \cdots y_{B_n} - y_{D_1'} y_{D_2'} y_{D_3} \cdots y_{D_n} \qquad (3)$$

also belongs to J_M.

By adding (1) and (3) and subtracting (2) we have that

$$y_{B_1} y_{B_2} y_{B_3} \cdots y_{B_n} - y_{D_1} y_{D_2} y_{D_3} \cdots y_{D_n}$$

belongs to J_M, as desired.

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Conjecture 1 For any matroid M, the toric ideal I_M has a Gröbner basis consisting of quadratics binomials.

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Remark : Conjectures 2 and 3 together imply White's conjecture.