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An algorithm for computing the multigraded Hilbert depth of a module

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Stanley depth

Let K be a field, $S = K[x_1, ..., x_n]$ be the polynomial ring in n variables, and M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element in M and Z a subset of $\{x_1, ..., x_n\}$. We denote by $uK[Z]$ the K-subspace of M generated by all elements uv where v is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $uK[Z] \subset M$ is called a *Stanley space of dimension* $|Z|$, if $uK[Z]$ is a free $K[Z]$-module.

A *Stanley decomposition* of M is a presentation of the $\mathbb{Z}^n$-graded K-vector space M as a finite direct sum of Stanley spaces

$$\mathscr{D} : M = \bigoplus_{i=1}^{m} u_i K[Z_i]$$

in the category of $\mathbb{Z}^n$-graded K-vector spaces.

In other words, each of the summands is a $\mathbb{Z}^n$-graded K-subspace of M and the decomposition is compatible with the $\mathbb{Z}^n$-grading, i.e. for each $a \in \mathbb{Z}^n$ we have $M_a = \oplus_{i=1}^{m}(u_i K[Z_i])_a$. The number

$$\text{sdepth } \mathscr{D} = \min \{|Z_i| : i = \overline{1, m}\}$$

is called the *Stanley depth of* $\mathscr{D}$.

The *Stanley depth* of M is defined to be

$$\text{sdepth } M = \max \{ \text{ sdepth } \mathscr{D} : \mathscr{D} \text{ is a Stanley decomposition of } M\}.$$

It is conjectured by Stanley that depth $M \leq$ sdepth M for all $\mathbb{Z}^n$-graded S-modules M.

Hilbert depth

Let M be a finitely generated graded $R = K[x_1, \dots, x_n]$-module. A *Hilbert decomposition* is a finite family

$$\mathfrak{D} = (S_i, s_i)_{i \in I}$$

such that $s_i \in \mathbb{Z}^n$, S_i is a graded K-algebra retract of R for each $i \in I$, and

$$M \cong \bigoplus_{i \in I} S_i(-s_i)$$

as a graded K-vector space.

A Hilbert decomposition carries the structure of an R-module and has a well-defined depth, which is called the *depth of the Hilbert decomposition* $\mathfrak{D}$ and will be denoted by $\operatorname{hdepth} \mathfrak{D}$. The *Hilbert depth* of a module M is

$$\max\{\operatorname{hdepth} \mathfrak{D} \mid \mathfrak{D} \text{ is a Hilbert decomposition of } M\}$$

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and will be denoted by $\operatorname{hdepth} M$.

Proposition(Cimpoeas) Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. If $\operatorname{sdepth}(M) = 0$ then $\operatorname{depth}(M) = 0$. Conversely, if $\operatorname{depth}(M) = 0$ and $dim_K(M_a) \leq 1$ for any $a \in \mathbb{Z}^n$, then $\operatorname{sdepth}(M) = 0$.

Computing Hilbert depth

Let $H_M(X) = \sum_{a \in \mathbb{N}^n} H(M,a) X^a$ be the *Hilbert series* of M and $g \in \mathbb{N}^n$ such that M is positively g-determined. Then the Hilbert series of M can be recovered from the polynomial

$$H_M(X)_{\preceq g} := \sum_{0 \preceq a \preceq g} H(M,a) X^a.$$

Given $a, b \in \mathbb{Z}^n$ such that $a \preceq b$, we set

$$Q[a,b](X) := \sum_{a \preceq c \preceq b} X^c$$

and call it the *polynomial induced by the interval* $[a,b]$.

We define a *Hilbert partition* of the polynomial $H_M(X)_{\preceq g}$ to be an expression

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i \in I_{\mathfrak{P}}} Q[a^i, b^i](X)$$

as a finite sum of polynomials induced by the intervals $[a^i, b^i]$ (the notation $I_{\mathfrak{P}}$ makes clear the dependency on $\mathfrak{P}$ and so the finiteness). In order to describe the Hilbert decomposition of M induced by the Hilbert partition $\mathfrak{P}$ of $H_M(X)_{\preceq g}$, we introduce the following notations. Let

$$\rho : \{0 \preceq a \preceq g\} \longrightarrow \mathbb{N}, \quad \rho(a) := |Z_a|,$$

and for $0 \preceq a \preceq b \preceq g$ we set

$$\mathscr{G}[a,b] = \{c \in [a,b] \mid c_j = a_j \text{ for all } j \in \mathbb{N} \text{ with } X_j \in Z_b\}.$$

Theorem The following statements hold:

1. Let $\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^r Q[a^i, b^i](X)$ be a Hilbert partition of $H_M(X)_{\preceq g}$. Then

$$\mathfrak{D}(\mathfrak{P}) : M \cong \bigoplus_{i=1}^{r} \Big(\bigoplus_{c \in \mathscr{G}[a^i, b^i]} K[Z_{b^i}](-c) \Big)$$

is a Hilbert decomposition of M. Moreover,

$$\operatorname{hdepth} \mathfrak{D}(\mathfrak{P}) = \min\{\rho(b^i) : \ i = 1, \ldots, r\}.$$

2. Let $\mathfrak{D}$ be a Hilbert decomposition of M. Then there exists a Hilbert partition $\mathfrak{P}$ of $H_M(X)_{\preceq g}$ such that

$$\operatorname{hdepth} \mathfrak{D}(\mathfrak{P}) \geq \operatorname{hdepth} \mathfrak{D}.$$

In particular, $\operatorname{hdepth} M$ can be computed as the maximum of the numbers $\operatorname{hdepth} \mathfrak{D}(\mathfrak{P})$, where $\mathfrak{P}$ runs over the finitely many Hilbert partitions of $H_M(X)_{\preceq g}$.

Corollary Let M a finitely generated multigraded R-module. Then

$$\operatorname{hdepth} M = \max\{\operatorname{hdepth} \mathfrak{D}(\mathfrak{P}) : \mathfrak{P} \text{ is a Hilbert partition of } H_M(X)_{\preceq g}\}.$$

Restricting the search

Let B be a subset of $\mathbb{N}^n$ and $0 \leq s \leq n$. We define two subsets of B,

$$B_{<s} := \{a \in B : \rho(a) < s\} \quad \text{and} \quad B_{\geq s} := \{a \in B : \rho(a) \geq s\}.$$

Let $a \in B_{<s}$. We define the set

$$B_{=s}(a) := \{x \in B_{\geq s} : a \preceq x,\ \rho(x) = s\}.$$

Theorem Assume $\operatorname{hdepth} M \geq s$. Then there exists a Hilbert partition

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^{r} Q[a^i, b^i](X)$$

such that if $\rho(a^i) < s$ then $b^i \in B_{=s}(a)$.

Problem Find an algorithm to compute the Stanley depth for finitely generated multigraded R-modules M with $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$.

Algorithm 1: Function which checks if hdepth $\geq s$ recursively

Data: $g \in \mathbb{N}^n$, $s \in \mathbb{N}$ and a polynomial $P(X) = H_M(X)_{\preceq g} \in \mathbb{N}[X_1, \ldots, X_n]$

Result: *true* if hdepth $M \geq s$

Boolean **CheckHilbertDepth**(g,s,P);

begin

  if $P \notin \mathbb{N}[X_1, \ldots, X_n]$ **then**

    return *false*;

  Container E =**FindElementsToCover**(g,s,P);

  if `size`$(E) = 0$ **then**

    return *true*;

  else

    for i=`begin`(E) **to** i=`end`(E) **do**

      Container $C[i]$:=**FindPossibleCovers**$(g,s,P,E[i])$;

      if `size`$(C[i]) = 0$ **then**

        return *false*;

      for j=`begin`$(C[i])$ **to** j=`end`$(C[i])$ **do**

        Polynomial $\tilde{P}(X) = P(X) - Q[E[i], C[i][j]](X)$;

        if **CheckHilbertDepth**$(g,s,\tilde{P})$=*true* **then**

          return *true*;

    return *false*;

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Do we have equality?

Answer No. If $n = 4, M = R^2, N = m$ then

$$\begin{aligned}\mathfrak{D}(M \oplus N) :& (1,0,0)K[X_1,X_2,X_3] \oplus (0,1,0)K[X_1,X_2,X_4] \oplus \\ & (0,0,X_1)K[X_1,X_3,X_4] \oplus (0,0,X_2)K[X_1,X_2,X_3] \oplus \\ & (0,X_3,X_3)K[X_1,X_3,X_4] \oplus (0,X_3,0)K[X_2,X_3,X_4] \oplus \\ & (X_4,0,X_4)K[X_1,X_2,X_4] \oplus (X_4,0,0)K[X_2,X_3,X_4] \oplus \\ & (0,X_1X_2X_3,0)K[X_1,X_2,X_3] \oplus (X_1X_3X_4,0,0)K[X_1,X_3,X_4] \oplus \\ & (0,0,X_1X_2X_4)K[X_1,X_2,X_4] \oplus (0,0,X_2X_3X_4)K[X_2,X_3,X_4] \oplus \\ & (X_1X_2X_3X_4,0,0)K[X_1,X_2,X_3,X_4] \oplus (0,X_1X_2X_3X_4,0)K[X_1,X_2,X_3,X_4] \\ & (0,0,X_1X_2X_3X_4)K[X_1,X_2,X_3,X_4].\end{aligned}$$

$$3 = \operatorname{sdepth}(M \oplus N) = \operatorname{hdepth}(M \oplus N) > \operatorname{Min}\{\operatorname{sdepth}(M), \operatorname{sdepth}(N)\} = 2.$$

Problem In the particular case where $I \subset R$ is a monomial ideal, does $\operatorname{sdepth}(R \oplus I) = \operatorname{sdepth} I$ hold?

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Answer No. If $n = 6, I = m$ then

$\mathfrak{D}(R \oplus I)$: $(X_5, X_5)K[X_3, X_4, X_5, X_6] \oplus (X_6, 0)K[X_3, X_4, X_5, X_6] \oplus$
$(1,0)K[X_1, X_2, X_3, X_4] \oplus (0, X_1X_2X_3)K[X_1, X_2, X_3, X_4] \oplus$
$(X_5, 0)K[X_1, X_2, X_4, X_5] \oplus (0, X_1X_4X_5)K[X_1, X_2, X_4, X_5] \oplus$
$(X_6, X_6)K[X_1, X_2, X_3, X_6] \oplus (0, X_1X_3X_6)K[X_1, X_2, X_3, X_6] \oplus$
$(X_1X_5X_6, 0)K[X_1, X_2, X_5, X_6] \oplus (0, X_1)K[X_1, X_2, X_5, X_6] \oplus$
$(X_2X_3X_5, 0)K[X_2, X_3, X_5, X_6] \oplus (0, X_2)K[X_2, X_3, X_5, X_6] \oplus$
$(X_1X_3X_5, 0)K[X_1, X_3, X_4, X_5] \oplus (0, X_3)K[X_1, X_3, X_4, X_5] \oplus$
$(X_1X_4X_6, 0)K[X_1, X_2, X_4, X_6] \oplus (0, X_4)K[X_1, X_2, X_4, X_6] \oplus$
$(X_2X_5X_6, 0)K[X_2, X_4, X_5, X_6] \oplus (0, X_2X_4X_5)K[X_2, X_4, X_5, X_6] \oplus$
$(X_1X_3X_4X_6, 0)K[X_1, X_3, X_4, X_6] \oplus (0, X_3X_4X_6)K[X_1, X_3, X_4, X_6] \oplus$
$(X_2X_3X_4X_5, 0)K[X_2, X_3, X_4, X_5] \oplus (0, X_2X_3X_4)K[X_2, X_3, X_4, X_5] \oplus$
$(X_2X_4X_6, 0)K[X_2, X_3, X_4, X_6] \oplus (0, X_2X_3X_4X_6)K[X_2, X_3, X_4, X_6] \oplus$

$$(X_1X_2X_3X_5,0)K[X_1,X_2,X_3,X_5]\oplus(0,X_1X_2X_3X_5)K[X_1,X_2,X_3,X_5]\oplus$$
$$(X_1X_3X_5X_6,0)K[X_1,X_3,X_5,X_6]\oplus(0,X_1X_3X_5X_6)K[X_1,X_3,X_5,X_6]\oplus$$
$$(X_1X_4X_5X_6,0)K[X_1,X_4,X_5,X_6]\oplus(0,X_1X_4X_5X_6)K[X_1,X_4,X_5,X_6]\oplus$$
$$(X_1X_2X_3X_4X_5,0)K[X_1,X_2,X_3,X_4,X_5]\oplus$$
$$(0,X_1X_2X_3X_4X_5)K[X_1,X_2,X_3,X_4,X_5]\oplus$$
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$$(0,X_1X_2X_3X_4X_5X_6)K[X_1,X_2,X_3,X_4,X_5,X_6].$$

$4=\operatorname{sdepth}(R\oplus I)=\operatorname{hdepth}(R\oplus I)>\operatorname{sdepth}(I)=\operatorname{hdepth}(I)=3.$

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