Problems

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Problems

Problem 1. Show that

$$A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 1 & -2 & 1 & -1 \\ 3 & 0 & 5 & 1 \\ 7 & -1 & 12 & 5 \end{pmatrix}$$

is a configuration matrix.

Problem 2. Let $A \in \mathbb{Z}^{d \times n}$. Then I_A is a principal ideal if and only if $\operatorname{rank} A = n - 1$.

Problem 3. Let $A = (3, 4, 5) \in \mathbb{Z}^{1 \times 3}$. Compute I_A.

Problem 4. Let $I \subset K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the ideal generated by a set $\mathcal{S}$ of 2-minors of the $2 \times n$-matrix $X = \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$. We denote by $[i, j]$ a 2-minor with rows i and j. Show that I is a prime ideal if and only $[n]$ is the disjoint union of sets $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that $\mathcal{S} = \bigcup_{i=1}^{k} \{[i, j] : \{i, j\} \subset \mathcal{S}_k\}$.

Problem 5. Let $\mathrm{char}(K) = 0$ and let $\mathbf{b} \in \mathbb{Z}^n$. Then $I = (f_{\mathbf{b}}) \subset S$ is a radical ideal.

Problem 6. Let $\mathbf{b}_1, \ldots, \mathbf{b}_r \in \mathbb{Z}^n$ be $\mathbb{Q}$-linearly independent vectors. Then $f_{\mathbf{b}_1}, \ldots, f_{\mathbf{b}_r}$ is a regular sequence.

Problem 7. Show that $(x^k - y^k, x^l - y^l) : (xy)^\infty = (x - y)$. Which is the smallest integer m with the property that $(x^k - y^k, x^l - y^l) : (xy)^m = (x - y)$?

Problem 8. Let $L \subset \mathbb{Z}^n$ be a lattice. Prove that height $I_L = \operatorname{rank} L$.

Problem 9. Let $\mathcal{B}$ be a basis of a lattice L for which $\mathbb{Z}^n/L$ is torsionfree. Then $I_\mathcal{B} = I_L$ if and only if $I_\mathcal{B}$ is a prime ideal.

Problem 10. Let $I \subset S$ be the ideal of adjacent 2-minors of a $m \times n$-matrix of indeterminates.

(a) Show that I is a radical ideal if and only if $m \leq 2$ or $n \leq 2$.

(b) Find a polynomial $f \in S \setminus I$ with $f^2 \in I$, if $m = n = 3$.

The ideals $L(P, Q)$ are pretty well studied. Less is known about the algebras $K[P, Q]$.

Problem 11. Show that all the algebras $K[P, Q]$ are normal (and hence CM).

Problem 12. For which P and Q does the defining ideal J_{PQ} of $K[P, Q]$ admit a quadratic Gröbner basis. Is the initial ideal of J_{PQ} squarefree for a suitable monomial order?

Problem 13. What is the projective dimension and the regularity of J_{PQ}? For $Q = [2]$ we have a Hibi ring and the answer is known.

Problem 14. Compute the graded Betti numbers of the defining ideal of a Hibi ring $K[L]$ - for example when L is a planar lattice.

Problem 15. Let $I \subset S$ be generated by a regular sequence . Show that S/I is not rigid. (Hint: First show than I/I^2 is a free SI-module)

Problem 16. Let $I \subset S$ be a graded ideal, and assume that K is a perfect field and that $R = S/I$ is a reduced CM ring. Then R is rigid if and only if $\Omega_{R/K} \otimes \omega_R$ is CM.

Problem 17. Let $I \subset S$ be a graded ideal, and assume that K is a perfect field and that $R = S/I$ is a 1-dimensional reduced Gorenstein ring. Then R is rigid if and only if $\Omega_{R/K}$ is torsionfree.

Problem 18. Find an inseparable monomial ideal which is not rigid.

Problem 19. Let G be a graph and G^* its whisker graph. Show that G^* is bi-CM if and only if G is a complete graph.

Problem 20. Which of the ideals $L(P, Q)$ is bi-CM?

Problem 21. Which of the matroidal ideals are inseparable?

Problem 22. Let $\mathfrak{m}$ be the graded maximal ideal of $S = K[x_1, \dots, x_n]$. Compute the module $T^1(S/\mathfrak{m}^2)$.

Problem 23. Let $I \subset \mathfrak{m}^2$ be a graded ideal with $\dim S/I = 0$. Do we always have that $T^1(R) \neq 0$?

Problem 24. Let $R = K[H]$ be a numerical semigroup ring. Show that $T^1(R)$ is module of finite length.

Problem 25. Compute the length of $T^1(R)$ when $R = K[t^{h_1}, t^{h_2}]$.