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Binomial Fibers and Indispensable Binomials

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Markov Bases of Lattice Ideals (arXiv)

Binomial Ideals

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. A binomial is a polynomial of the form $x^{\mathbf{u}} - \lambda x^{\mathbf{v}}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ and $\lambda \in \mathbb{K} \setminus \{0\}$. A binomial ideal is an ideal generated by binomials. We say that the ideal I of R is a *pure binomial ideal* if I is generated by *pure difference binomials*, i.e. binomials of the form $x^{\mathbf{u}} - x^{\mathbf{v}}$ with $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$.

Generating sets of Binomial Ideals: Main problems

Let $B_1, \ldots, B_s$ be binomials in $R = \mathbb{K}[x_1, \ldots, x_n]$.

Questions

Let $I = \langle B_1, \ldots, B_s \rangle$.

1. *What are the invariants of I, in terms of its generating sets?*
2. *Are there indispensable binomials among the B_i?*
3. *What about the monomials that appears as the terms of binomials that generate I. Are there indispensable among those?*

The *indispensable elements* are *present* in every generating set of I.

Binomial Ideals and Fibers

Given a binomial ideal I, consider the equivalence relation:

$$\mathbf{u} \sim_I \mathbf{v} \text{ if } x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \in I, \text{ for some } \lambda \neq 0$$

(in Dickenstein, Matusevich, Miller, (Math. Z.) 2010). For each such equivalence class, we get a *fiber* on the set of the monomials of R. We let the fiber of $x^{\mathbf{u}}$, $F_{x^{\mathbf{u}}}$ to be the set

$$F_{x^{\mathbf{u}}} := \{x^{\mathbf{v}} : \quad \mathbf{u} \sim_I \mathbf{v}\}.$$

The *system of fibers* of I is the set of I-fibers and it forms a partition of the monomials of R. (Fibers when multiplied by monomials end up inside other fibers).

An Example of a system of fibers

$I = \langle y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8\rangle$.
Finite fibers:

- singletons: $F_{x^a y^b}$, where (a, b) has to be in the region with corners $(7, 0)$, $(5, 1)$, $(4, 2)$, $(3, 2)$, $(2, 3)$, $(2, 4)$, $(1, 5)$, $(0, 7)$
- $|F_{xy^6}| = |F_{x^6y}| = |F_{xy^7}| = |F_{x^7y}| = 2$, $|F_{x^3y^3}| = 3$,

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Three infinite fibers. Let $S = \mathbb{N}(-1, 2) + \mathbb{N}(2, -1)\}$.

- $F_{x^4y^3} = \{x^ay^b : \ (a, b) \in \mathbb{N}^2 \cap ((4, 3) + S)\}$
- $F_{x^3y^4} = \{x^ay^b : \ (a, b) \in \mathbb{N}^2 \cap ((3, 4) + S)\}$
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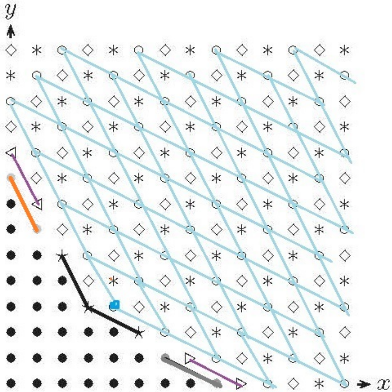
Example

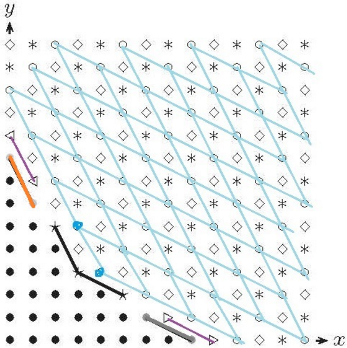
$$I = (y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8).$$



blue represents the fiber of x^4y^3.

Example

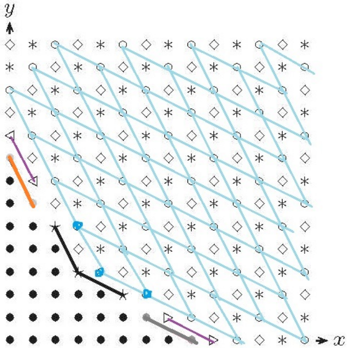
$$I = (y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8).$$



$x^4y^3 - x^3y^5 = x(x^3y^3 - x^2y^5) \in I$, so $x^4y^3 \equiv x^3y^5$
or equivalently
$x^4y^3 = x(x^3y^3) \equiv x(x^2y^5) = x^3y^5$.

Example

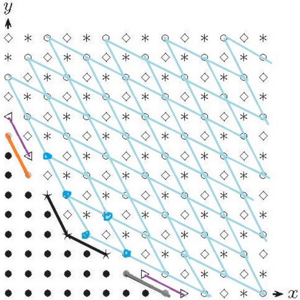
$$I = (y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8).$$



(the black fiber underneath has been pushed to the right with multiplication by x and gives blue monomials).

Example

$$I = (y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8).$$

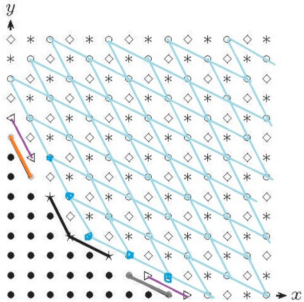


But we can also push the black fiber up by y^2 and get three monomials, one of them already blue. So all of them have to be blue.

$x^5y^4 = (x^5y^2)y^2 \equiv x^3y^5 \equiv x^4y^3$

Example

$$I = (y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8).$$



we can also push gray up by y and get to a blue monomial $x^6y^2 = (x^6y)y \equiv (x^8)y = x^8y$, etc.

When is a partition of the sets of monomials a system of fibers?

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. We denote by n the set of monomials of R including $1 = x^0$, where $x^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$.

Theorem

Let $\mathbf{F}$ be a partition of $\mathbb{T}^n$. There exists a binomial ideal I such that $\mathbf{F}$ is the set of I-fibers if and only if for any $\mathbf{u} \in \mathbb{N}^n$ and any $F \in \mathbf{F}$ there exists a $G \in \mathbf{F}$ such that $x^{\mathbf{u}} F \subset G$. Moreover, if F is a system of fibers, then there exists a pure difference binomial ideal whose system of fibers is F.

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Idea of the proof: If $\mathbf{F} = (F_i : \ i \in \Lambda)$ satisfies the conditions consider the ideal generated by the set $\{x^{\mathbf{u}} - x^{\mathbf{v}} : \ \mathbf{u}, \mathbf{v} \in F_i, i \in \Lambda\}$ and show that the system of its fibers is F.

What do we know about systems of fibers.

1. If L is a lattice and $L \cap \mathbb{N}^n = \{\mathbf{0}\}$, then the fibers are finite.
2. In general, I may have finite and infinite fibers.
3. The fibers depend on the characteristic.
4. If I contains monomials, then the monomials (of I) form a fiber, the monomial fiber.
5. We can define an equivalence relation on the system of fibers and order the equivalence classes.
6. Any descending chain of such equivalence classes terminates since R is Noetherian.

Ordering the fibers.

Let I be a binomial ideal, F, G two I-fibers. Then

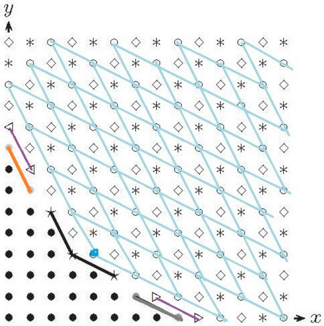
$$F \equiv G \text{ if } \exists \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ s.t. } x^{\mathbf{u}} F \subset G \text{ and } x^{\mathbf{v}} G \subset F.$$

We denote by $\overline{F}$, the equivalence class of F. We set $\overline{F} \leq \overline{G}$ if there exists $\mathbf{u} \in \mathbb{N}^n$ such that $x^{\mathbf{u}} F \subset G$. We write $\overline{F} < \overline{G}$ if $\overline{F} \leq \overline{G}$ and $\overline{F} \neq \overline{G}$.

- If I is a lattice ideal, then any two equivalence classes of fibers have the same cardinality (—arxiv).
- If a fiber is finite then it is the only element in its equivalence class.
- If I contains monomials and the monomial fiber exists, then its equivalence class contains exactly one element and it is maximal.

Example of equivalence classes.

$$I = \langle y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8 \rangle.$$



What are the equivalence classes? How are they ordered?

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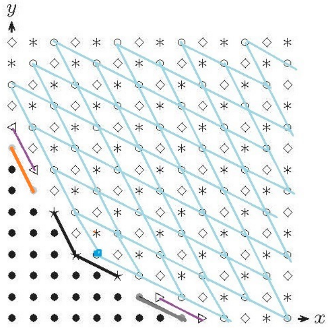

What are the equivalence classes? How are they ordered?
The infinite fibers are equivalent.
orange < purple < blue.

Invariants of Binomial Ideals

Theorem

Let I be a binomial ideal, S a binomial generating set of I and F an I-fiber. Then the set $\{\overline{F_B} : B \in S\}$ is an invariant of I.

Idea of the proof: Consider the ideals $I_{<\overline{F}} = (B \in I : B$ binomial, $\overline{F}_B < \overline{F})$ and $I_{\leq\overline{F}} = (B \in I : \overline{F}_B \leq \overline{F})$. Show that these ideals are determined by S, i.e. $I_{<\overline{F}} = (B \in S : \overline{F}_B < \overline{F})$ and similarly for $I_{\leq\overline{F}}$.

Example

$I = \langle y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8\rangle$. All fibers of this generating set are finite. Thus every generating set of I must have elements that determine exactly three fibers: F_{xy^6}, $F_{x^3y^3}$, F_{x^6y}.

Fibers and Invariants of Binomial Ideals

If I is a lattice ideal I and S_1, S_2 are two minimal binomial generating sets of I of minimal cardinality, then

$$\{\overline{F_B}: \ B \in S_1\} = \{\overline{F_B}: \ B \in S_2\}$$

where the equality holds for the *multisets* (i.e. sets together with the multiplicities of their elements), (—arXiv).

Question

Are there binomials that appear in every minimal generating set? What about the monomial terms of the generating sets. Are they somehow unique? Can we determine the monomials that will always appear?

(Indispensable binomials, Indispensable monomials)

Indispensable monomials

Let I be a binomial ideal and let M_I be the ideal generated by all monomials $x^{\mathbf{u}}$ such that $0 \neq x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \in I$, for some $\mathbf{v} \in \mathbb{N}^n$.

Indispensable monomials

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Theorem

If $I = (x^{\mathbf{u}_1} - \lambda_1 x^{\mathbf{v}_1}, \ldots, x^{\mathbf{u}_s} - \lambda_s x^{\mathbf{v}_s})$ with $\lambda_1, \ldots, \lambda_s \in \mathbb{K} \setminus \{0\}$ then $M_I = (x^{\mathbf{u}_1}, x^{\mathbf{v}_1}, \ldots, x^{\mathbf{u}_s}, x^{\mathbf{v}_s})$. The indispensable monomials of I are precisely the elements of $G(M_I)$.

(Generalization of a result of Katsabekis, Ojeda (P.J.M.) 2014).

Example

$I = \langle y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8 \rangle$. Thus

$$M_I = \langle y^8, xy^6, x^2y^5, x^3y^3, x^5y^2, x^6y, x^8 \rangle.$$

Since $|G(M_I)| = 7$, every binomial generating set of I must consist of at least 4 elements.

Indispensable binomials

Theorem

Let I be a binomial ideal. The binomial $0 \neq x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \in I$ $(\lambda \neq 0)$ is indispensable if and only if the fiber F of $\mathbf{u}$ consists of exactly two monomials and $x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \notin I_{<\overline{F}}$.

Example

Let $I = \langle y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8 \rangle$. The binomials correspond to three minimal fibers. However, as we saw $|F_{xy^6}| = |F_{x^6y}| = 2$ while $|F_{x^3y^3}| = 3$. Thus the indispensable binomials of I are $y^8 - xy^6$ and $x^6y - x^8$.

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But do we have to compute the fibers to find the indispensable binomials?

A NEW Algorithm for the Computation of Indispensable binomials

Previous methods (for toric ideals when $L \cap \mathbb{N}^n = \{0\}$):

- the algorithm in (Ohsugi,Hibi, 2005) implies computation of $n!$ reduced Gröbner bases with respect to the lexicographic orders,
- the algorithm in (–,Katsabekis, Thoma, 2007) implies the computation of the minimal elements in the set of I-fibers, via the computation of one Gröbner basis,
- the algorithm in (Ojeda, Vigneron-Tenorio, 2010) implies computation of n reduced Gröbner basis with respect to n degree reverse lexicographic orders.

A NEW Algorithm for the Computation of Indispensable binomials

The new algorithm is a polynomial time algorithm.

- It basically checks for divisibility of the monomial terms of the given binomial generating set and counts how many times they appear.
- The binomial generating set needs not be minimal.
- In particular no Gröbner bases computation needs to be involved.

The Algorithm for the Computation of Indispensable binomials

What we know so far: The binomial $0 \neq x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \in I$ is indispensable if and only if the fiber F of $\mathbf{u}$ consists of exactly two monomials and $x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \notin I_{<\overline{F}}$.

We let $\mathrm{Supp}(x^{\mathbf{u}} - \lambda x^{\mathbf{v}}) = \{x^{\mathbf{u}}, x^{\mathbf{v}}\}$. If S is a finite set of binomials, we let $\mathcal{F}(S)$ be the graph with vertex set $\cup_{B \in S} \mathrm{Supp}(B)$ and edges $\{x^{\mathbf{u}}, x^{\mathbf{v}}\}$ whenever $x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \in S$, up to a nonzero scalar multiplication.

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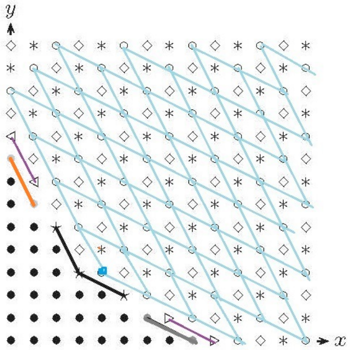
Theorem

Let S be a binomial minimal generating set of I, The binomial $B = x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \in S$ is indispensable if and only if $x^{\mathbf{u}}, x^{\mathbf{v}} \in G(M_I)$, $\{x^{\mathbf{u}}, x^{\mathbf{v}}\}$ is a simple edge of $\mathcal{F}(S)$ and a connected component of $\mathcal{F}(S)$.

Example

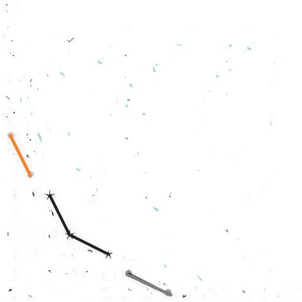
Let $I = \langle y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8 \rangle$.



From the study of the fibers we know that there are exactly two indispensable binomials.

Example

Let $I = \langle y^8 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8 \rangle$.



$$G(M_I) = \{y^8, xy^6, x^2y^5, x^3y^3, x^5y^2, x^6y, x^8\}.$$

From the graph $\mathcal{F}(S)$ and the minimal generating set of M_I we get immediately that there exactly two indispensable binomials.

The Algorithm for the Computation of Indispensable binomials

Algorithm 1 Computing the indispensable binomials of a binomial ideal I

Input: $F = \{B_1, \dots, B_s\} \subseteq \mathbb{K}[X]$, with $B_i = x^{\mathbf{u}_i} - \lambda_i x^{\mathbf{v}_i}$ for $i \in [s]$, where $\lambda_1, \dots, \lambda_s \in \mathbb{K} \setminus \{0\}$.

Output: $F' \subset F$, the set of indispensable binomials of $I = (B_1, \dots, B_s)$.

1: Compute $G(M_I)$, a subset of $\{x^{\mathbf{u}_1}, x^{\mathbf{v}_1}, \dots, x^{\mathbf{u}_s}, x^{\mathbf{v}_s}\}$, and set $T = \{i : \{x^{\mathbf{u}_i}, x^{\mathbf{v}_i}\} \subset G(M_I)\}$.

2: If $T = \emptyset$ then $F' = \emptyset$.

3: Otherwise, for every $i \in T$ check whether $x^{\mathbf{u}_i} \in \mathrm{Supp}(B_j)$ or $x^{\mathbf{v}_i} \in \mathrm{Supp}(B_j)$ for some $j \neq i$.

4: $F' = \{x^{\mathbf{u}_i} - \lambda_i x^{\mathbf{v}_i} : i \in T, x^{\mathbf{u}_i} \notin \mathrm{Supp}(B_j), x^{\mathbf{v}_i} \notin \mathrm{Supp}(B_j), \text{ for all } j \neq i\}$.

Testing if the ideal is generated by indispensable binomials

Algorithm 2 Testing whether a binomial ideal is generated by indispensable binomials

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Input: $F = \{B_1, \ldots, B_s\} \subseteq \mathbb{K}[X]$, a set of binomials generating I.
Output: Is I generated by indispensable binomials? YES or NO

1: Compute $S \subset F$, a set of minimal generators for I.
2: Compute $G(M_I)$ from S.
3: If $|G(M_I)| = 2|S|$ then I is generated by indispensable binomials, otherwise not.

Application In (Aoki, Takemura, (A.N.Z.J.Stat) 2003) it was shown that a toric ideal is generated by indispensable binomials. We recover this result as follows: first, using a computer algebra program, we compute a generating set for $I_{\mathcal{A}_{333}}$: using CocoA we see that it has cardinality 114. We minimize to get a generating set of cardinality 81 and finally we compute $|G(M_{I_{\mathcal{A}_{333}}})|$, which is 162.

Graver Basis

Let I be a pure binomial ideal. A binomial $0 \neq x^{\mathbf{u}} - x^{\mathbf{v}} \in I$ is called a *primitive binomial* of I if there exists no other binomial $0 \neq x^{\mathbf{u}'} - x^{\mathbf{v}'} \in I$ such that $x^{\mathbf{u}'}$ divides $x^{\mathbf{u}}$ and $x^{\mathbf{v}'}$ divides $x^{\mathbf{v}}$. The set of all primitive binomials of I is called the *Graver basis* of I, and denoted by $\mathrm{Gr}(I)$.

Theorem

Let I be a pure binomial ideal. Every binomial in the universal Gröbner basis of I is contained in $\mathrm{Gr}(I)$. In particular, $\mathrm{Gr}(I)$ is a generating set for the ideal I. Moreover, $\mathrm{Gr}(I)$ is a finite set.

Proof of the theorem and a Corollary

Idea of proof: Universal Gröbner basis $\subset$ of $\mathrm{Gr}(I)$ as for toric ideals.
To show that $\mathrm{Gr}(I)$ is a finite set look at the set
$\{x^{\mathbf{u}}y^{\mathbf{v}},\ x^{\mathbf{v}}y^{\mathbf{u}} :\ x^{\mathbf{u}} - x^{\mathbf{v}} \in \mathrm{Gr}(I)\}$ which has no divisibility relations and use the following lemma:

Lemma

If a set of monomials has no divisibility relations then it is necessarily finite.

$\square$

If I is a pure binomial ideal, consider
$\Lambda(I) := (x^{\mathbf{u}}y^{\mathbf{v}} - x^{\mathbf{v}}y^{\mathbf{u}} :\ x^{\mathbf{u}} - x^{\mathbf{v}} \in \mathrm{Gr}(I))$.

Corollary

$\Lambda(I)$ *is generated by indispensable binomials.*

Caution The minimal generating set of $\Lambda(I)$ is not necessarily a Graver basis of $\Lambda(I)$ or the universal Gröbner basis of $\Lambda(I)$.