Resolutions of letterplace and co-letterplace ideals

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A very short invitation to (squarefree) monomial ideals

- **Commutative algebra:** *free resolutions* of modules are important because they encode structure information (as an example, think of presenting Abelian groups by generators and relations). Betti numbers are numerical invariants tied to minimal free resolutions.
- **Combinatorics:** *simplicial complexes* are collections of simplices (i.e. points, segments, triangles, tetrahedra...) glued together by their faces. They are in one-to-one correspondence with squarefree monomial ideals by the Stanley-Reisner correspondence.
- **Algebraic topology:** *homology* is a set of invariants attached to a simplicial complex (and, more generally, to a topological space). Homotopy equivalent spaces have the same homology.

Tools like Hochster's formula allow a fruitful interplay between commutative algebra, combinatorics and algebraic topology!

Posets

- For the whole talk P and Q will be finite partially ordered sets.
- A map of sets ϕ from P to Q is said *isotone* if it respects the order, i.e.
$$p \leq p' \Rightarrow \phi(p) \leq \phi(p').$$
We will denote by $Hom(P, Q)$ the set of isotone maps.
- The set $Hom(P, Q)$ can be given a poset structure in the following way:
$$\psi \leq \phi \text{ if and only if } \psi(p) \leq \phi(p) \text{ for all } p \in P.$$
- A *poset ideal* $\mathcal{J}$ of P is a subset of P "closed below", i.e.
$$p \in \mathcal{J}, p' \leq p \Rightarrow p' \in \mathcal{J}.$$

Letterplace and co-letterplace ideals

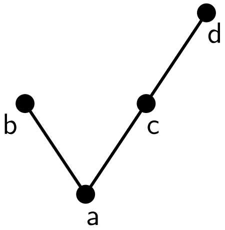
- [Fløystad-Greve-Herzog 2015] Let S be the polynomial ring over $\Bbbk$ with variables $x_{p,q}$, where (p,q) ranges in $P \times Q$. Then we can associate with each isotone map ϕ (or, more precisely, with its graph $\Gamma\phi$) the squarefree monomial

$$m_{\Gamma\phi} := \prod_{p \in P} x_{p,\phi(p)}.$$

  The monomial ideal generated by all possible $m_{\Gamma\phi}$ is denoted by $L(P,Q)$.
- The cases where P or Q is the totally ordered poset $[n]$ deserve special names:
  - $L([n],P)$ is called a *letterplace ideal*;
  - $L(P,[n])$ is called a *co-letterplace ideal*.

An example

As an example, let us consider the letterplace ideal $L([3], P)$, where P is the poset represented by the Hasse diagram below.



$$\begin{aligned} L(3,P) = (&x_{1,a}x_{2,a}x_{3,a},\ x_{1,a}x_{2,a}x_{3,b},\ x_{1,a}x_{2,a}x_{3,c},\ x_{1,a}x_{2,a}x_{3,d} \\ &x_{1,a}x_{2,b}x_{3,b},\ x_{1,a}x_{2,c}x_{3,c},\ x_{1,a}x_{2,c}x_{3,d},\ x_{1,a}x_{2,d}x_{3,d} \\ &x_{1,b}x_{2,b}x_{3,b},\ x_{1,c}x_{2,c}x_{3,c},\ x_{1,c}x_{2,c}x_{3,d},\ x_{1,c}x_{2,d}x_{3,d},\ x_{1,d}x_{2,d}x_{3,d}) \end{aligned}$$

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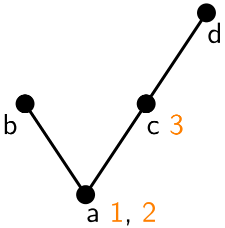

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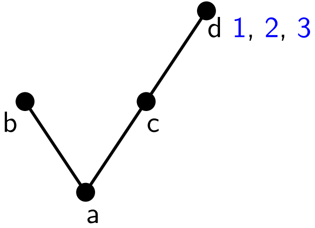

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Why are (co-)letterplace ideals interesting?

- Let R be a poset and let $\phi : [n] \times P \to R$ be a map of posets. Let $L^{\phi}(n, P)$ be the (not necessarily squarefree) monomial ideal generated by the image of $L(n, P)$ under ϕ.
- [Fløystad-Greve-Herzog 2015] If ϕ satifies an extra technical condition, then $\Bbbk[x_{im\phi}]/L^{\phi}(n, P)$ can be obtained by quotienting out $\Bbbk[x_{[n]\times P}]/L(n, P)$ by a regular sequence of linear forms (actually, differences of variables).
- As a consequence, $L^{\phi}(n, P)$ and $L(n, P)$ have the same graded Betti numbers!
- Some objects that can be expressed in the form $L^{\phi}(n, P)$: classical initial ideals of determinantal and ladder determinantal ideals, multichain ideals.
- An analogous result holds for co-letterplace ideals, but the proof is different (and more delicate)!

First results

[Ene-Herzog-Mohammadi 2011, Fløystad-Greve-Herzog 2015]

- $L(n, P)$ and $L(P, n)$ are Alexander dual.
- $L(n, P)$ is Cohen-Macaulay and hence $L(P, n)$ has a linear resolution. Actually, $L(P, n)$ even has linear quotients and hence $L(n, P)$ is the Stanley-Reisner ideal of a shellable complex: more about this later!
- Given $L(P, n)$, we can also consider only the isotone maps that lie in a poset ideal $\mathcal{J}$. The ideal $L(\mathcal{J})$ still has linear quotients.

Question

Can we say something more precise on Betti numbers (and, more generally, minimal resolutions) of letterplace and co-letterplace ideals?

Letterplace ideals

Let us start from letterplace ideals $L(n, P)$. Is there a combinatorial way of characterizing their Betti numbers?

[D.-Fløystad-Nematbakhsh 2016]

- **Bad news**: in general, Betti numbers of letterplace ideals (even $L(2, P)$) do depend on the characteristic. One can "simulate the homology of any simplicial complex" inside a suitable $L(2, P)$ (see also Dalili and Kummini 2014).
- **Good news**: the data for the Betti numbers of $L(n, P)$ in a fixed multidegree R can be retrieved by splitting the problem into several $L(2, Q_i^{(R)})$, where the $Q_i^{(R)}$'s are suitable posets depending on the choice of R.
- Further **good news**: if P is a rooted tree, Betti numbers of $L(n, P)$ do not depend on the characteristic and there exists a recursive procedure to compute them.

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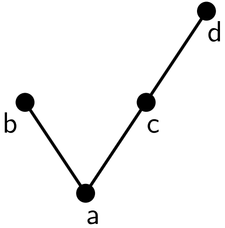
An example

As an example, let us compute $\beta_{i,R}(L(3,P))$, where P is the poset represented by the Hasse diagram below and $R = \{1a, 2b, 2c, 3b, 3d\}$.



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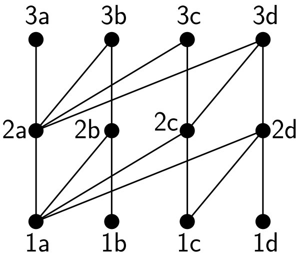
An example

- Step 1: represent $L(n, P)$ as a multipartite graph as below.



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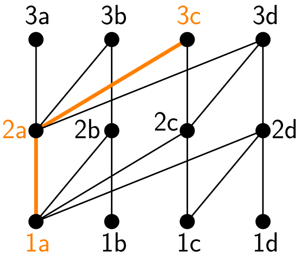

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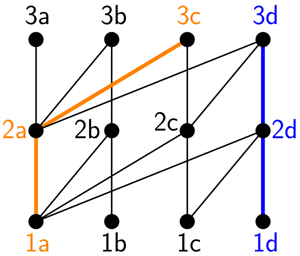

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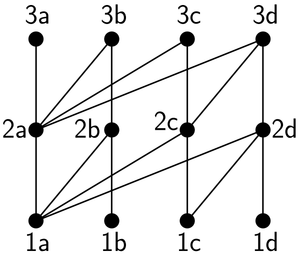
- Step 2: take the subgraph induced by the chosen multidegree R.



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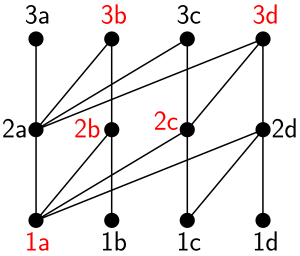

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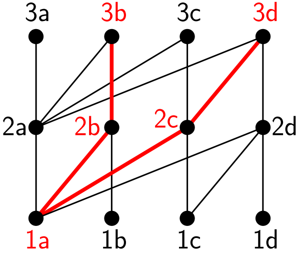

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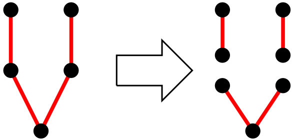
- Step 3a: duplicate all the “floors” in the middle of the graph, obtaining $n-1$ bipartite graphs.



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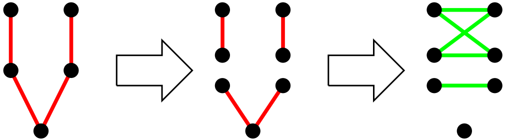
- Step 3a: duplicate all the "floors" in the middle of the graph, obtaining $n-1$ bipartite graphs.
- Step 3b: use the edge ideals of the bipartite graphs as Stanley-Reisner ideals.



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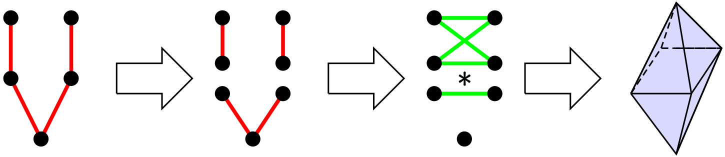
- Step 3a: duplicate all the “floors” in the middle of the graph, obtaining $n-1$ bipartite graphs.
- Step 3b: use the edge ideals of the bipartite graphs as Stanley-Reisner ideals.
- Step 3c: take the join of the $n-1$ complexes obtained in this way.



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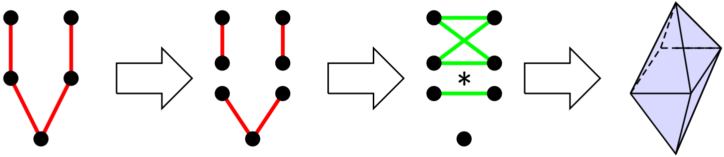

We claim that the last object in the picture above (which is homeomorphic to a 2-sphere) is homotopy equivalent to $\Delta|_R$, where Δ is the complex whose Stanley-Reisner ideal is $L(3, P)$.

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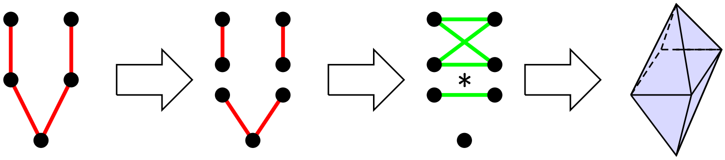

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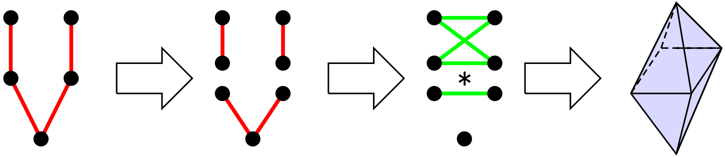

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Therefore, by Hochster's formula,

$$\beta_{i,R}(L(3,P)) = \dim_{\Bbbk} \widetilde{H}_{|R|-i-2}(\Delta|_R; \Bbbk) = \dim_{\Bbbk} \widetilde{H}_{3-i}(\mathbb{S}^2; \Bbbk).$$

As a consequence, $\beta_{1,R}(L(3,P)) = 1$ and $\beta_{i,R}(L(3,P)) = 0$ for all $i \neq 1$.

Co-letterplace ideals

For co-letterplace ideals the situation is way nicer!

Ene, Herzog and Mohammadi (2011) proved that $L(P, n)$ has linear quotients and admits a regular decomposition function. As a consequence, the iterated mapping cone technique yields a minimal free resolution for $L(P, n)$: to write down this resolution, though, one needs to compute some data depending on the linear quotients.

We already know that, for any poset ideal $\mathcal{J}$ in $Hom(P, n)$, $L(\mathcal{J})$ has linear quotients.

Question

Is there a way to compute the minimal free resolution of $L(\mathcal{J})$? Can we do it without investigating decomposition functions?

The resolution

Let $\mathcal{J}$ be a poset ideal in $Hom(P, n)$. We will denote by $\mathcal{J}^c$ the complement of $\mathcal{J}$ inside $Hom(P, n)$. Moreover, let B be the squarefree monomial ideal generated by all possible $x_{p,i}x_{p',j}$ with $p < p'$ and $i > j$.

Theorem (D.-Fløystad-Nematbakhsh 2016)

Let $\mathbb{F}_\bullet$ be the minimal free resolution of $L(\mathcal{J})$ over $S = \Bbbk[x_{P\times[n]}]$. Then

$$F_i = \bigoplus_{\substack{m_A \text{ squarefree monomial in } L(\mathcal{J}) \\ m_A \notin B + L(\mathcal{J}^c),\ i=|A|}} S(-A)$$

and, denoting by e_A the generator of multidegree A, the differential is

$$e_A \mapsto \sum_{a\in A_2} (-1)^{\alpha(a,A)} x_a e_{A\setminus a},$$

where A_2 and $\alpha(a, A)$ are "objects that can be read off instantly from A".

The ingredients of the proof

The proof of the last theorem uses several algebraic tools, notably:

- [Yanagawa 2000, Miller 2000, Römer 2001] the definition of *squarefree module* and the extension of the concept of Alexander duality to an exact contravariant functor $\mathrm{SqfrMod} \to \mathrm{SqfrMod}$ (note that the "new" Alexander duality sends I_Δ to $S/I_{\Delta^\vee}$);
- [Yanagawa 2004] the fact that, if M is a squarefree Cohen-Macaulay module, then the resolution of the Alexander dual M^* is given by an explicit complex depending on the canonical module of M.

Question

Can one extend these techniques to more general classes of ideals with linear resolution?

A combinatorial byproduct: PL spheres

Incidentally, each poset $\mathcal{J}$ in $Hom(P, n)$ explicitly gives us a simplicial PL ball and hence, as its boundary, a simplicial PL sphere! There are essentially two reasons behind this.

- **Combinatorics:** by a well-known criterion (Bing, Danaraj-Klee, Björner), a pure shellable complex can be recognized as a PL ball if all its codimension 1 faces are contained in at most two facets and containment in one facet occurs.
- **Algebra:** (Stanley, Bruns-Herzog) a CM complex Δ is a homology ball precisely when the canonical module $\omega_{\Bbbk[\Delta]}$ embeds into $\Bbbk[\Delta]$ as a proper multigraded ideal.

Generalized Bier spheres

Note that we have obtained a very simple way to generate explicitly **a lot** of (Stanley-Reisner ideals of) PL homology spheres!
When $n = 2$, P is an antichain and $\mathcal{J}$ is the full $Hom(P, 2)$, the spheres we are talking about are the so-called *Bier spheres*. These objects were introduced by Bier in 1992 and then investigated deeply by Björner, Paffenholz, Sjöstrand and Ziegler in 2004.

- For purely numerical reasons, most Bier spheres do not admit a convex realization. Using the same argument, most generalized Bier spheres are not convex as well.
- Bier spheres satisfy a stronger version of the g-conjecture. What about generalized Bier spheres?

Thank you for your attention!