Connectedness and regularity for dual graphs of projective curves

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EMS SUMMER SCHOOL ON MULTIGRADED ALGEBRA AND APPLICATIONS

August 17-24, Moieciu de Sus, Romania

joint work with Bruno Benedetti and Matteo Varbaro

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$$\sqrt{I} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s \quad \text{and} \quad \mathrm{height}(\mathfrak{p}_i) = \mathrm{height}(I), \; i = 1, \dots, s.$$

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- $R = S/I$, $X = \mathrm{Proj}(R)$,

$$X_{red} = X_1 \cup \ldots \cup X_s,$$

where $X_i = \mathrm{Proj}(S/\mathfrak{p}_i)$ are the irreducible components,

$$\mathrm{codim}(X_i) = \mathrm{codim}(X) = \mathrm{height}(I), \; i = 1, \ldots, s.$$

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Algebraic ($R = S/I$)	Geometric ($X = \mathrm{Proj}(R)$)
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$\{i, j\} \in E \Updownarrow \mathrm{height}(\mathfrak{p}_i + \mathfrak{p}_j) = \mathrm{height}(I) + 1$	$\{i, j\} \in E \Updownarrow \dim(X_i \cap X_j) = \dim(X) - 1$

Example: dual graph of subspace arrangements

$X \subset \mathbb{P}^n_k$ subspace arrangement:

$$X = X_1 \cup \ldots \cup X_s,$$

X_i projective subspace of dimension d, $\forall i = 1, \ldots, s$.

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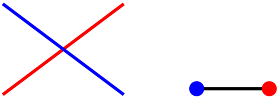
$$G(X) = ([s], E), \text{ where } \{i, j\} \in E \Leftrightarrow \dim(X_i \cap X_j) = d - 1.$$

(Here "dim" is the dimension as projective spaces.)



Example: dual graph of simplicial complexes

Δ pure simplicial complex on $n+1$ vertices with facets $\{F_1, \dots, F_s\}$:

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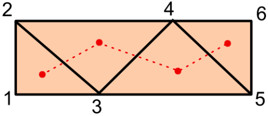

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Here $G(C) = ([s], E)$, where $\{i, j\} \in E \Leftrightarrow C_i \cap C_j \neq \emptyset$.

The inclusions are strict

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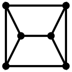

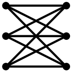

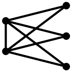

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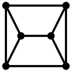

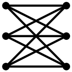

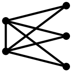

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$$G(X) = G(C).$$

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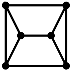
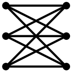
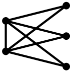

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Hence:

$$\left\{\begin{array}{c}\text{dual graphs}\\ \text{of subspace}\\ \text{arrangements}\end{array}\right\} = \left\{\begin{array}{c}\text{dual graphs}\\ \text{of line}\\ \text{arrangements}\end{array}\right\} ; \left\{\begin{array}{c}\text{dual graphs}\\ \text{of projective}\\ \text{varieties}\end{array}\right\} = \left\{\begin{array}{c}\text{dual graphs}\\ \text{of projective}\\ \text{curves}\end{array}\right\}$$

Measures of connectedness

Two ways of quantifying the connectedness of a simple graph G on the vertex set $[s]$ are provided by the following invariants:

- the diameter of G:

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If G is r-connected, every vertex in G has at least r neighbours.

G is said r-regular when every vertex of G has exactly r neighbours.

Castelnuovo-Mumford regularity

Consider a minimal free resolution of R as S-module:

$$\mathbb{F}_. : 0 \to F_p \to \dots F_j \to F_{j-1} \to \dots \to F_1 \to F_0 \to R \to 0.$$

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The (Castelnuovo-Mumford) regularity of R is:

$$\operatorname{reg}(R) := \min\{r \mid F_j \text{ is generated in degrees } \leq r + j,\ \forall j\}.$$

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Remark

When $I = (f_1, \ldots, f_h)$ is a complete intersection ideal (i.e. $h = \text{height}(I)$), $deg(f_i) = d_i$ and $R = S/I$, we have:

$$\text{reg}(R) = d_1 + \ldots + d_h - h.$$

Good properties on $S/I \Rightarrow$ Better connectedness on $G(I)$

Theorem (Hartshorne 1962)

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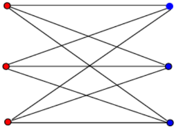
Theorem (Benedetti, Varbaro 2015)

$X = X_1 \cup \ldots \cup X_s \subset \mathbb{P}^n$ arithmetically Gorenstein (e.g. complete intersection) subspace arrangement of regularity r.
Then $G(X)$ is $r-1$-connected, and hence

$$\operatorname{diam}(G(X)) \leq \lfloor \frac{s-2}{r} \rfloor + 1.$$

Example: lines on a smooth quadric

If $Q \subseteq \mathbb{P}^3$ is a smooth quadric, and X is the union of p lines of a ruling of Q, and q of the other ruling, then $G(X)$ is the complete bipartite graph $K_{p,q}$.



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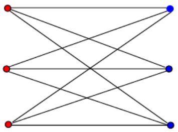

Figure : $K_{3,3}$

One can check that $X \subseteq \mathbb{P}^3$ is a complete intersection (of Q and an union of p planes) if and only if $p = q$. In this case

- $\operatorname{reg} X - 1 = p$.
- $G(X)$ is p-connected.
- $G(X)$ is p-regular.

Example: 27 lines on a cubic

Let $Z \subseteq \mathbb{P}^3$ be a smooth cubic, and $X = \bigcup_{i=1}^{27} X_i$ be the union of all the lines on Z.

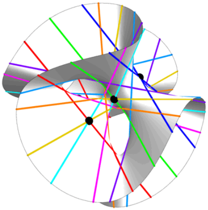

Figure : Clebsch's cubic: $x_0^3 + x_1^3 + x_2^3 + x_3^3 = (x_0 + x_1 + x_2 + x_3)^3$

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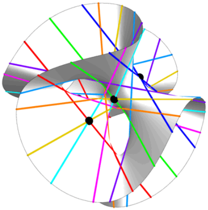

Figure : Clebsch's cubic: $x_0^3 + x_1^3 + x_2^3 + x_3^3 = (x_0 + x_1 + x_2 + x_3)^3$

X is a complete intersection (of Z and an union of 9 planes). In this case:

- $\operatorname{reg} X - 1 = 10$.
- $G(X)$ is 10-connected.
- $G(X)$ is 10-regular.

Examples in higher degrees

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- For every d, this line arrangement X on Z has dual graph $G(X)$ consisting in $3d$ complete graphs K_d, and each pair of K_d is connected by a complete matching.
- $G(X)$ is $4d - 2$-regular.
- X is a complete intersection between Z and an union of $3d$ planes and hence $\mathrm{reg}(X) = (4d - 2) - 1$.

The two notions of regularity coincides!

Theorem (Benedetti, D., Varbaro)

$X \subset \mathbb{P}^n$ arithmetically Gorenstein (e.g. complete intersection) line arrangement with regularity r having planar singularities.
Then $G(X)$ is $r-1$-regular.

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Remark

- If no three lines of the arrangement meet at the same point, the hypothesis of having only planar singularities is fulfilled.

The hypothesis of having planar singularities is necessary

Example

- Let $Y := Proj(S/J) \subset \mathbb{P}^{n-1}$, where $J = (f_1, \dots, f_d)$ is a complete intersection of $n-1$ polynomials of degree d.
- The cone $X \subset \mathbb{P}^n$ of Y is an arrangement of d^{n-1} lines in $\mathbb{P}^n$ with

$$\operatorname{reg} X = (n-1)d - n + 2.$$

- Since all lines in X pass through the origin, $G(X)$ is the complete graph, so it is $(d^{n-1} - 1)$-regular and

$$d^{n-1} - 1 >> ((n-1)d - n + 2) - 1.$$

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There are also examples of complete intersection line arrangements with non-planar singularities whose dual graph is not even regular!

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Corollary

Suppose to have the line arrangement

$$X = L_1 \cup \ldots \cup L_{de} \subset \mathbb{P}^3,$$

consisting of $d \cdot e$ lines.
If the lines lie on two surfaces of degree d and e without common component and one of them is smooth, then each line meets exactly

$$d + e - 2$$

of the others.

What about other projective curves?

Theorem (Benedetti, Bolognese, Varbaro 2015)

$X \subset \mathbb{P}^n$ arithmetically Gorenstein curve of regularity r.
If every primary component of X has regularity $\leq R$, then

$$G(X) \text{ is } \left\lfloor \frac{r+R-2}{R} \right\rfloor - \text{connected.}$$

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Conjecture (Benedetti, Varbaro 2014)

If $X \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay and $I_X \subset S$ is generated by quadrics, then

$$\operatorname{diam}(G(X)) \leq \operatorname{height}(I_X) (= \operatorname{codim}(X)).$$

About the conjecture

Proposition

The conjecture is true up to codimension 4 for aGorenstein reduced.

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Without the assumption "generated by quadrics" the conjecture is false already for line arrangements in codimension 2 (while for simplicial complex is always true for codim $= 2, 3$).

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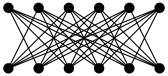
Example (Schläfli double six)

There is a sub-arrangement X of the 27 lines on a smooth cubic having the following dual graph:



X is a complete intersection of a cubic and a quartic.
Yet, $\text{diam}(G(X)) = 3$.