Lecture 1: Lattice ideals and lattice basis ideals

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Outline

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Toric ideals

Let K be a field. We denote by $S = K[x_1, \dots, x_n]$ the polynomial ring in the variables $x_1, \dots, x_n$. A binomial belonging to S is a polynomial of the form $u - v$, where u and v are monomials in S.

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A binomial ideal is an ideal of S generated by binomials. Any binomial ideal is generated by a finite number of binomials.

An important class of binomial ideals are the so-called toric ideals.

In order to define toric ideals we let $A = (a_{ij})_{\substack{1 \le i \le d \\ 1 \le j \le n}}$ be a $d \times n$-matrix of integers and let

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{dj} \end{pmatrix}, \quad 1 \le j \le n$$

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We write $\mathbb{Z}^{d \times n}$ for the set of $d \times n$-matrices $A = (a_{ij})_{\substack{1 \le i \le d \\ 1 \le j \le n}}$ with each $a_{ij} \in \mathbb{Z}$.

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As usual $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$ denotes the inner product of the vectors $\mathbf{a} = (a_1, \ldots, a_n)^t$ and $\mathbf{b} = (b_1, \ldots, b_n)^t$. Here $\mathbf{c}^t$ denotes transpose of a vector $\mathbf{c}$.

A matrix $A = (a_{ij})_{\substack{1 \le i \le d \\ 1 \le j \le n}} \in \mathbb{Z}^{d \times n}$ is called a configuration matrix if there exists $\mathbf{c} \in \mathbb{Q}^d$ such that

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For example, $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ is a configuration matrix, while $(a_1, \ldots, a_n) \in \mathbb{Z}^{1 \times n}$ is a configuration matrix if and only if $a_1 = a_2 = \ldots = a_n \neq 0$.

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Now let $T = K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ be the Laurent polynomial ring over K in the variables $t_1, \ldots, t_n$, and let $A \in \mathbb{Z}^{d \times n}$ with column vectors $\mathbf{a}_j$.

We define a K-algebra homomorphism

$$\pi : S \to T \quad \text{with} \quad x_j \mapsto \mathbf{t}^{\mathbf{a}_j}.$$

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Proposition: Let $A \in \mathbb{Z}^{d \times n}$. Then $\dim K[A] = \operatorname{rank} A$.

Proof. Let $K(A)$ be the quotient field of $K[A]$. Then the Krull dimension of $K[A]$ is equal to the transcendence degree $\mathrm{trdeg}(K(A)/K)$ of $K(A)$ over K.

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Let $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be a $\mathbb{Q}$-basis of integer vectors of V. Then $m = \operatorname{rank} A$ and $K(A) = K(\mathbf{t}^{\mathbf{b}_1}, \ldots, \mathbf{t}^{\mathbf{b}_m})$. The desired result will follow once we have shown that the elements $\mathbf{t}^{\mathbf{b}_1}, \ldots, \mathbf{t}^{\mathbf{b}_m}$ are algebraically independent over K.

To see this, let $F \in K[y_1, \dots, y_m]$ be a polynomial with $F(\mathbf{t}^{\mathbf{b}_1}, \dots, \mathbf{t}^{\mathbf{b}_m}) = 0$. Say, $F = \sum_{\mathbf{c}} a_{\mathbf{c}} \mathbf{y}^{\mathbf{c}}$ with $a_{\mathbf{c}} \in K$.

Then

$$0 = \sum_{\mathbf{c}} a_{\mathbf{c}} \mathbf{t}^{c_1 \mathbf{b}_1 + \cdots + c_m \mathbf{b}_m}.$$

Since the vectors $\mathbf{b}_1, \dots, \mathbf{b}_m$ are linearly independent it follows that the monomials $\mathbf{t}^{c_1 \mathbf{b}_1 + \cdots c_m \mathbf{b}_m}$ are pairwise distinct. This implies that $F = 0$. $\square$

Given a column vector

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

belonging to $\mathbb{Z}^n$, we introduce the binomial $f_{\mathbf{b}} \in S$, defined by

$$f_{\mathbf{b}} = \prod_{b_i > 0} x_i^{b_i} - \prod_{b_j < 0} x_j^{-b_j}.$$

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Note that $f_{\mathbf{b}} = \mathbf{x}^{\mathbf{b}^+} - \mathbf{x}^{\mathbf{b}^-}$, where

$$b_i^+ = \begin{cases} b_i, & \text{if } b_i \geq 0, \\ 0, & \text{if } b_i < 0, \end{cases} \quad \text{and} \quad b_i^- = \begin{cases} 0, & \text{if } b_i > 0, \\ -b_i, & \text{if } b_i \leq 0. \end{cases}$$

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Theorem. Any toric ideal is a binomial ideal. More precisely, let $A \in \mathbb{Z}^{d \times n}$. Then I_A is generated by the binomials $f_{\mathbf{b}}$ with $\mathbf{b} \in \mathbb{Z}^n$ and $A\mathbf{b} = 0$.

Proof. We first show that I_A is a binomial ideal. Let $f \in \operatorname{Ker} \pi$ with $f = \sum_u \lambda_u u$, $\lambda_u \in K$ and each u a monomial in S. We write $f = \sum_{\mathbf{c}} f^{(\mathbf{c})}$, where $f^{(\mathbf{c})} = \sum_{u,\, \pi(u) = \mathbf{t}^{\mathbf{c}}} \lambda_u u$.

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It follows that

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and hence $\sum_{u,\ \pi(u)=\mathbf{t}^{\mathbf{c}}} \lambda_u = 0$ for all $\mathbf{c}$. Thus if $f^{(\mathbf{c})} \neq 0$ and $u \in \operatorname{supp}(f^{(\mathbf{c})})$, then $f^{(\mathbf{c})} = \sum_{v \in \operatorname{supp}(f^{(\mathbf{c})})} \lambda_v (v - u)$.

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Finally, let $f_{\mathbf{b}} \in S$. Then $\pi(f_{\mathbf{b}}) = \mathbf{t}^{A\mathbf{b}^+} - \mathbf{t}^{A\mathbf{b}^-}$. Hence $f_{\mathbf{b}} \in \operatorname{Ker}\pi$ if and only if $A\mathbf{b}^+ = A\mathbf{b}^-$, and this is the case if and only if $A\mathbf{b} = 0$.
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(c) I_A is a graded ideal.

Proof. We only proof (b)$\Leftrightarrow$ (c). The binomials $f_{\mathbf{b}}$ with $A\mathbf{b} = 0$ generate I_A. Thus I_A is graded if and only if all $f_{\mathbf{b}}$ are homogeneous. This is the case if and only if $\sum_{i=1}^n b_i = 0$ for all $\mathbf{b}$ with $A\mathbf{b} = 0$. $\square$.

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A subgroup L of $\mathbb{Z}^n$ is called a lattice. Recall from basic algebra that L is a free abelian group of rank $m \leq n$. The binomial ideal $I_L \subset S$ generated by the binomials $f_{\mathbf{b}}$ with $\mathbf{b} \in L$ is called the lattice ideal of L.

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Consider for example, the lattice $L \subset \mathbb{Z}^3$ with basis $(1,1,1),(1,0,-1)$. Then $\mathbf{b} \in L$ if and only if $A\mathbf{b} = 0$ where $A = (1-2,1)$. Thus in this case we have that I_L is a toric ideal.

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On the other hand, any toric ideal is a lattice ideal.

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Not all lattice ideals are toric ideals. The simplest such example is the ideal I_L for $L = 2\mathbb{Z} \subset \mathbb{Z}$. Here $I_L = (x^2 - 1)$. If I_L would be a toric ideal it would be a prime ideal. But $x^2 - 1 = (x+1)(x-1)$, and so I_L is not a prime ideal.

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We have the following general result:

Theorem. Let $L \subset \mathbb{Z}^n$ be a lattice. The following conditions are equivalent:

(a) the abelian group $\mathbb{Z}^n/L$ is torsionfree;

(b) I_L is a prime ideal;

The equivalent conditions hold, if and only if I_L is a toric ideal.

We only indicate the proof of (a) $\Longrightarrow$ (b): Since $\mathbb{Z}^n/L$ is torsion free, there exists an embedding $\mathbb{Z}^n/L \subset \mathbb{Z}^d$ for some d. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the canonical basis of $\mathbb{Z}^n$. Then for $i = 1, \ldots, n$, $\mathbf{e}_i + L$ is mapped to $\mathbf{a}_i \in \mathbb{Z}^d$ via this embedding. It follows that $\sum_{i=1}^n b_i\mathbf{a}_i = 0$ if and only if $\mathbf{b} = (b_1, \ldots, b_n)^t \in L$. In other words, $\mathbf{b} \in L$ if and only if $A\mathbf{b} = 0$, where A is the matrix whose column vectors are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Therefore, I_L is the toric ideal of A, and hence a prime ideal. $\square$

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Let I and J be two ideals. The saturation of I with respect to J is the ideal $I : J^\infty$, where by definition $I : J^\infty = \bigcup_k (I : J^k)$.

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Consider the polynomial ring $T = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ over K

in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then

$T/(x_1y_1 - 1, \ldots, x_ny_n - 1) \simeq S_x$, and hence
$T/(I, x_1y_1 - 1, \ldots, x_ny_n - 1)T \simeq S_x/IS_x$.

Therefore, $IS_x \cap S = (I, x_1y_1 - 1, \ldots, x_ny_n - 1)T \cap S$, from which it follows that $I : (\prod_{i=1}^{n} x_i)^{\infty}$ is a binomial ideal. $\square$

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We claim that $L \subset \mathbb{Z}^n$ is a lattice. Indeed, if $\mathbf{b} \in L$ then $uf_{\mathbf{b}} \in I$ for some monomial u and hence $uf_{-\mathbf{b}} = -uf_{\mathbf{b}} \in I$. This shows that $-\mathbf{b} \in L$.

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$$\begin{aligned} (uf_{\mathbf{b}})(vf_{\mathbf{c}}) &= uv(wf_{\mathbf{b}+\mathbf{c}} - \mathbf{x}^{\mathbf{b}^-} f_{\mathbf{c}} - \mathbf{x}^{\mathbf{c}^-} f_{\mathbf{b}}) \\ &= uvwf_{\mathbf{b}+\mathbf{c}} - \mathbf{x}^{\mathbf{b}^-} u(vf_{\mathbf{c}}) - \mathbf{x}^{\mathbf{c}^-} v(uf_{\mathbf{b}}). \end{aligned}$$

It follows from this equation that $\mathbf{b} + \mathbf{c} \in L$. This proves the claim.

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It follows from this equation that $\mathbf{b}+\mathbf{c} \in L$. This proves the claim.

In the next step one shows that $I : (\prod_{i=1}^n x_i)^\infty = I_L$. $\square$

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Proof. We only need to show that $I_L : (\prod_{i=1}^n x_i)^\infty \subset I_L$. Let $f \in I_L : (\prod_{i=1}^n x_i)^\infty$. We may assume that f is a binomial, and we may further assume that $f = f_{\mathbf{b}}$ for some $\mathbf{b} \in \mathbb{Z}^n$.

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We want to show that $\mathbf{b} \in L$. Since $f_{\mathbf{b}} \in I_L : (\prod_{i=1}^n x_i)^\infty$, it follows that $1 - \mathbf{x}^{\mathbf{b}} \in I_L S_x$ where $x = \prod_{i=1}^n x_i$. Observe that $I_L S_x$ is generated by the binomials $1 - \mathbf{x}^{\mathbf{c}}$ with $\mathbf{c} \in L$. Therefore,$S_x / I_L S_x$ is isomorphic to the group ring $K[\mathbb{Z}^n / L]$ which admits the K-basis consisting of the elements of the group $G = \mathbb{Z}^n / L$.

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Let $g = \mathbf{b} + L$. Then $1 - g = 0$ in $K[\mathbb{Z}^n / L]$ because $1 - \mathbf{x}^{\mathbf{b}} \in I_L S_x$. This implies that $\mathbf{b} + L = 0 + L$, and hence $\mathbf{b} \in L$, as desired. $\square$

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Corollary. Let $I \subset S$ be a binomial ideal. Then I is a lattice ideal if and only if $I : (\prod_{i=1}^n x_i)^\infty = I$.

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Theorem Let $L \subset \mathbb{Z}^n$ be a lattice and let t be the maximal order of a torsion element of $\mathbb{Z}^n/L$. If $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > t$, then I_L is a radical ideal.

Lattice basis ideals

Let $L \subset \mathbb{Z}^n$ be a lattice and let $\mathcal{B} = \mathbf{b}_1, \ldots, \mathbf{b}_m$ be a $\mathbb{Z}$-basis of L. The ideal $I_{\mathcal{B}}$ is called a lattice basis ideal of L.

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In general, $I_{\mathcal{B}} \neq I_L$. Consider for example, $A = (3, 4, 5) \in \mathbb{Z}^{1\times 3}$. The toric ideal I_A is the lattice ideal of the lattice L with basis $\mathcal{B} = (2, 1, -2), (1, -2, 1)$. Then $I_{\mathcal{B}} = (x^2y - z^2, xz - y^2)$, while I_L also contains the binomial $x^3 - yz$.

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Corollary. Let $\mathcal{B}$ be a basis of the lattice L. Then $I_{\mathcal{B}} : (\prod_{i=1}^n x_i)^\infty = I_L$.

Proof. There exists a lattice $L' \subset \mathbb{Z}^n$ such that $I_{\mathcal{B}} : (\prod_{i=1}^{n} x_i)^\infty = I_{L'}$. Since $\mathcal{B} \subset L'$ it follows that $L \subset L'$.

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On the other hand, $I_{\mathcal{B}} \subset I_L$. Thus, $I_{L'} = I_{\mathcal{B}} : (\prod_{i=1}^n x_i)^\infty \subset I_L : (\prod_{i=1}^n x_i)^\infty = I_L$. This shows that $L' \subset L$, and hence $L' = L$. $\square$

Proof. There exists a lattice $L' \subset \mathbb{Z}^n$ such that $I_\mathcal{B} : (\prod_{i=1}^n x_i)^\infty = I_{L'}$. Since $\mathcal{B} \subset L'$ it follows that $L \subset L'$.

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We fix a field K, and let $X = (x_{ij})$ be an $(m \times n)$-matrix of indeterminates. The ideal of all 2-minors of X is a prime ideal and and hence may be viewed as a toric ideal, or as a lattice ideal I_L for the lattice $L \subset \mathbb{Z}^{m \times n}$ with lattice basis $\mathcal{B}$ consisting of the vectors

$$e_{ij} + e_{i+1,j+1} - e_{i,j+1} - e_{i+1,j}, \quad 1 \leq i \leq m-1, 1 \leq j \leq n-1.$$

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The ideal $I_{\mathcal{B}}$ is called the ideal of adjacent minors of X. It has first been studied by Hoşten and Sullivant.

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Let I be an ideal in a Noetherian ring. Then there exists an integer k such that $(\sqrt{I})^k \subset I$. The smallest integer with this property is called the index of nilpotency, denoted $\mathrm{nilpot}(I)$.

Theorem. (Ene, H, Hibi and Qureshi) I be the ideal of adjacent 2-minors of the generic $(m \times n)$-matrix X, and let $n = 4k + p$ and $n = 4l + q$ with $0 \leq p, q < 4$. Then

$$\mathrm{nilpot}(I) \geq (k + \lfloor \frac{p}{3} \rfloor)(l + \lfloor \frac{q}{3} \rfloor) + 1 \approx \frac{mn}{16}.$$

Problems

Problem 1. Show that

$$A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 1 & -2 & 1 & -1 \\ 3 & 0 & 5 & 1 \\ 7 & -1 & 12 & 5 \end{pmatrix}$$

is a configuration matrix.

Problem 2. Let $A \in \mathbb{Z}^{d \times n}$. Then I_A is a principal ideal if and only if $\operatorname{rank} A = n - 1$.

Problem 2. Let $A = (3, 4, 5) \in \mathbb{Z}^{1 \times 3}$. Compute I_A.

Problem 3. Let $I \subset K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the ideal generated by a set $\mathcal{S}$ of 2-minors of the $2 \times n$-matrix $X = \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$. We denote by $[i, j]$ a 2-minor with rows i and j. Show that I is a prime ideal if and only $[n]$ is the disjoint union of sets $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that $\mathcal{S} = \bigcup_{i=1}^{k} \{[i, j] : \{i, j\} \subset \mathcal{S}_k\}$.

Problem 4. Let $\mathrm{char}(K) = 0$ and let $\mathbf{b} \in \mathbb{Z}^n$. Then $I = (f_{\mathbf{b}}) \subset S$ is a radical ideal.

Problem 5. Let $\mathbf{b}_1, \ldots, \mathbf{b}_r \in \mathbb{Z}^n$ be $\mathbb{Q}$-linearly independent vectors. Then $f_{\mathbf{b}_1}, \ldots, f_{\mathbf{b}_r}$ is a regular sequence.

Problem 6. Show that $(x^k - y^k, x^l - y^l) : (xy)^\infty = (x - y)$. Which is the smallest integer m with the property that $(x^k - y^k, x^l - y^l) : (xy)^m = (x - y)$?

Problem 7. Let $L \subset \mathbb{Z}^n$ be a lattice. Prove that height $I_L = \operatorname{rank} L$.

Problem 6. Let $\mathcal{B}$ be a basis of a lattice L for which $\mathbb{Z}^n/L$ is torsionfree. Then $I_{\mathcal{B}} = I_L$ if and only if $I_{\mathcal{B}}$ is a prime ideal.

Problem 7. Let $I \subset S$ be the ideal of adjacent 2-minors of a $m \times n$-matrix of indeterminates.
(a) Show that I is a radical ideal if and only if $m \leq 2$ or $n \leq 2$.
(b) Find a polynomial $f \in S \setminus I$ with $f^2 \in I$, if $m = n = 3$.