Lecture 2: Ideals and Algebras defined by Isotone Maps between Posets

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Outline

Hibi rings

The category of posets and ideals attached to graphs of isotone maps

Alexander duality for such ideals

The K-algebra $K[P, Q]$ given by the posets P and Q

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Fix a field K and let L be a finite distributive lattice. The Hibi ring $K[L]$ is the K-algebra generated over K by the elements $\alpha \in L$ with defining relations

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Furthermore, the defining ideal of a Hibi ring has a quadratic Gröbner basis and hence is a Koszul algebra.

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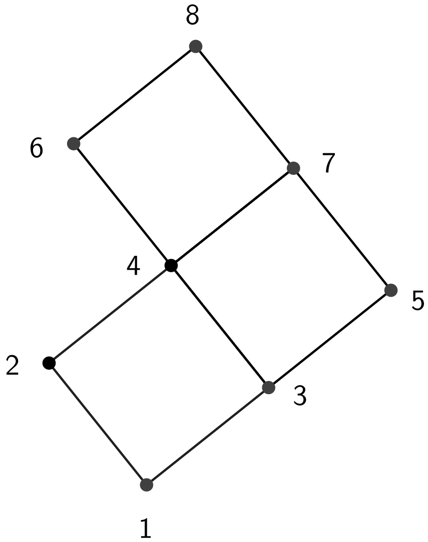
Birkhoff: $L \simeq I(P)$.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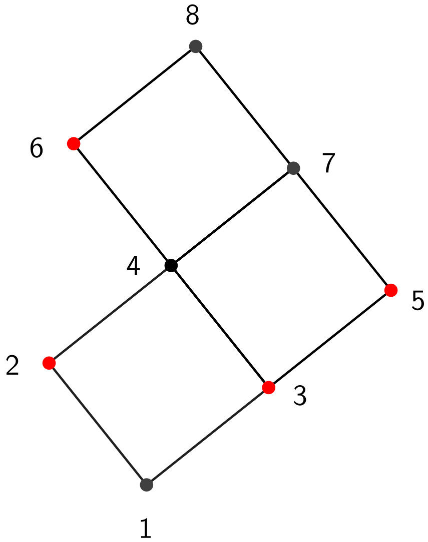
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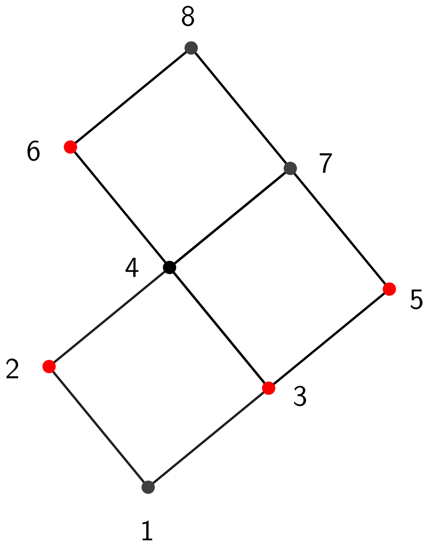
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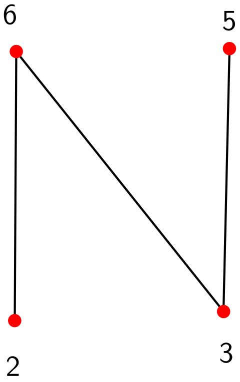
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$K[L]$ is Gorenstein if and only if P is pure (that is, all maximal chains in P have the same length).

Alternatively, the Hibi ring of L has a presentation

$$K[L] \simeq K[\{s \prod_{p \in \alpha} t_p : \alpha \in L\}] \subset T,$$

where $T = K[s, \{t_p \mid p \in P\}]$ is the polynomial ring in the variables s and t_p.

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Let $\hat{P}$ be the poset obtained from P by adding the elements $-\infty$ and ∞ with $\infty > p$ and $-\infty < p$ for all $p \in P$.

We denote by $\mathcal{T}(\hat{P})$ the set of integer valued functions

$$v : \hat{P} \to \mathbb{N}$$

with $v(\infty) = 0$ and $v(p) < v(q)$ for all $p > q$.

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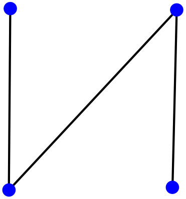
These are the strictly order reversing functions on $\hat{P}$.



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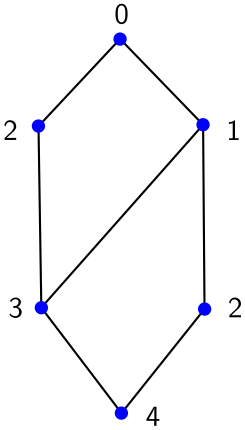

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By using a result of Richard Stanley, Hibi showed that the monomials

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Let J_L denote the defining ideal of the Hibi ring $K[L]$.

Theorem. (Ene, H, Saeedi Madani) Let L be a finite distributive lattice and P the poset of join irreducible elements of L. Then

$$\operatorname{reg} J_L = |P| - \operatorname{rank} P.$$

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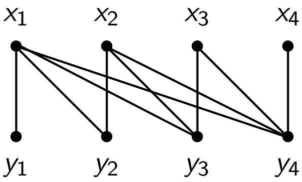

Let $\mathcal{P}$ be the category of finite posets.

- Objects: finite posets
- Morphisms: isotone maps (i.e. order preserving maps)

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$\mathrm{Hom}(P, Q)$, the set of isotone maps from P to Q, is itself a poset. We denote by $[n]$ the totally ordered poset $\{1 < 2 < \cdots < n\}$ on n elements. Then

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Now the theorem of Birkhoff, can be rephrased as follows: Let P be the subposet of join irreducible elements of the distributive lattice L. Then

$$L \simeq \mathrm{Hom}(P, [2])$$

Fløystad, H, Greve introduced the ideals

$$L(P,Q) = (\prod_{p \in P} x_{p,\varphi(p)} : \ \varphi \in \mathrm{Hom}(P,Q))$$

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The generators of $L(P,[n])$ are in bijection to the chains

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n = P$$

of poset ideals.

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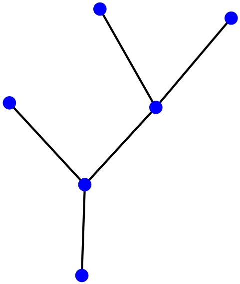
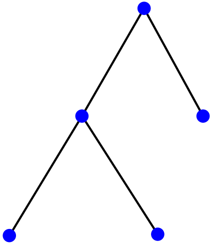
P is (co)-rooted if for all incomparable $p_1, p_2 \in P$ there is no $p \in P$ with $p > p_1, p_2$ ($p < p_1, p_2$).





Theorem. (H, Shikama, Qureshi) $L(P,Q)^\vee = L(Q,P)^\tau$ if and only if P or Q is connected and one of the following conditions hold:

(a) Both, P and Q are rooted;

(b) Both, P and Q are co-rooted;

(c) P is connected and Q is a disjoint union of chains;

(d) Q is connected and P is a disjoint union of chains;

(e) P or Q is a chain.

In the recent paper "Algebraic properties of ideals of poset homomorphisms" Juhnke-Kubitzke, Katthän and Saeedi Madani show for large subclasses of the ideals $L(P, Q)$ when they are Buchsbaum, Cohen-Macaulay, Gorenstein and when they have a linear resolution.

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In their paper "Resolutions of letter place ideals of posets" the same authors develop some topological results to compute their multigraded Betti numbers, and to give structural results on these Betti numbers.

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- Let I be the initial ideal of the ideal of s-minors of an $(n+s-1) \times (m+s-1)$-matrix of indeterminates. Then I is obtained from $L([s], [m] \times [n])$ by reduction modulo a regular sequence which identifies variables.

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- Strongly stable ideals.

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Theorem. (a) (Fløystad, H, Greve) Any monomial ideal I generated by a subset of the monomial generators of $L(P, Q)$ is inseparable.
(b) (Altmann, Bigdeli, H, Dancheng Lu) The ideals $L(P, Q)$ are rigid if and only if no two elements of P are comparable.

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An inseparable monomial ideal I which specializes to a monomial ideal J is called a separated model of J. So the ideals $L(P, Q)$ are separated models of many monomial ideals.

The K-algebra $K[P, Q]$

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What is the Krull dimension of $K[P, Q]$?

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What is the Krull dimension of $K[P, Q]$?

Theorem. (Bigdeli, Hibi, H, Shikama, Qureshi) Let P and Q be finite posets. Then $\dim K[P, Q] = |P|(|Q| - s) + rs - r + 1$, where r is the number of connected components of P and s is the number of connected components of Q.

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Conjecture. For any poset P and Q, the defining ideal of the K-algebra $K[P, Q]$ has a squarefree initial ideal.

Assuming the conjecture is true, the algebras $K[P, Q]$ are all normal by a theorem of Sturmfels, and then by a theorem of Hochster they are also Cohen-Macaulay.

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- $K[[2], P]$ is the edge ring of a bipartite graph. The binomials corresponding to the cycles of the graph form a Gröbner basis. These generators have a squarefree initial ideal.
- $Q = [n]$.

Theorem. (Bigdeli, Hibi, H, Shikama, Qureshi) Let P be the chain and suppose that each connected component of Q is either rooted or a co-rooted. Then the defining toric ideal of $K[P, Q]$ admits a quadratic Gröbner basis and a squarefree initial ideal.

The ideals $L(P, Q)$ are pretty well studied. Less is known about the algebras $K[P, Q]$.

Problem 1: Show that all the algebras $K[P, Q]$ are normal (and hence CM).

Problem 2: For which P and Q does the defining ideal J_{PQ} of $K[P, Q]$ admit a quadratic Gröbner basis. Is the initial ideal of J_{PQ} squarefree for a suitable monomial order?

Problem 3: What is the projective dimension and the regularity of J_{PQ}? For $Q = [2]$ we have a Hibi ring and the answer is known.

Problem 4: Compute the graded Betti numbers of the defining ideal of a Hibi ring $K[L]$ - for example when L is a planar lattice.