Lecture 3: Deformations and separations

Jürgen Herzog
Universität Duisburg-Essen

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Deformations

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Thus we obtain a commutative diagram of standard graded K-algebras

$$\begin{array}{ccc} C & \longrightarrow & A \\ \uparrow & & \uparrow \\ B & \longrightarrow & K. \end{array}$$

Let $I \subset B$ be a graded ideal. Then $B \to C$ induces the flat homomorphism $B/I \to C/IC$, and hence induces the deformation

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Any surjective K-algebra homomorphism $B \to K[\epsilon]$ induces a deformation of A with basis $K[\epsilon]$.

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Lemma. $K[\epsilon] \to C$ is flat if and only if $0 :_C \epsilon = \epsilon C$.

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We have the exact sequence

$$\cdots \xrightarrow{\epsilon} K[\epsilon] \xrightarrow{\epsilon} K[\epsilon] \longrightarrow K[\epsilon]/(\epsilon) \longrightarrow 0.$$

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Whenever there is a deformation $B \to C$ of A with $B \neq k$, then there is also an infinitesimal deformation, induced by a surjective K-algebra homomorphism.

Thus, if there is no infinitesimal deformation, then there cannot by any other deformation.

An infinitesimal deformation always exists. For example

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However this is a trivial deformation.

More generally we say that C is a trivial deformation of A with basis B, if $C \simeq A \otimes_K B$ as a B-algebra, and this isomorphism induces the identity on A modulo $\mathfrak{m}_B$.

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An infinitesimal deformation of A which is induced by a deformation of A with basis $K[t]$ (the polynomial ring), is called unobstructed.

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Let $J \subset S[\epsilon]$ be a graded ideal, and let $C = S[\epsilon]/J$ be a potential infinitesimal deformation of S/I.

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Let $J \subset S[\epsilon]$ be a graded ideal, and let $C = S[\epsilon]/J$ be a potential infinitesimal deformation of S/I.

Proposition: Let $I = (f_1, \ldots, f_m)$. Then $J = (f_1 + g_1\epsilon, \ldots, f_m + g_m\epsilon)$ and $K[\epsilon] \to S[\epsilon]/J$ is flat if and only $\varphi : I \to S/I$ with $f_i \mapsto g_i + I$ is a is a well-defined S-module homomorphism.

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Let $C = S[\epsilon]/J$ be an infinitesimal deformation of S/I. Then this deformation is trivial if and only if there a $K[\epsilon]$-automorphism $\varphi : S[\epsilon] \to S[\epsilon]$ such that $\varphi(J) = IS[\epsilon]$.

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Then

$$\begin{aligned}
\varphi(\prod_{i=1}^n x_i^{a_i}) &= \prod_{i=1}^n (x_i + \partial x_i \epsilon)^{a_i} = \prod_{i=1}^n (x_i^{a_i} + a_i x_i^{a_i - 1} \partial x_i \epsilon) \\
&= \prod_{i=1}^n x_i^{a_i} + \sum_{i=1}^n a_i x_i^{a_i - 1} \partial x_i \prod_{j \neq i} x_j^{a_j} \epsilon \\
&= \prod_{i=1}^n x_i^{a_i} + \partial(\prod_{i=1}^n x_i^{a_i}) \epsilon.
\end{aligned}$$

Since φ and ∂ are K-linear, it follows that $\varphi(f_i) = f_i + \partial f_i \epsilon$ for all i. Therefore, $\varphi^{-1}(J) = IS[\epsilon]$. □

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As a consequence of our considerations so far we see the following: if we consider the natural map $\delta^* : \operatorname{Der}_K(S) \to I^*$ which assigns to $\partial \in \operatorname{Der}_K(S)$ the element $\delta^*(\partial)$ with

$$\delta^*(\partial)(f_i) = \partial f_i + I,$$

then the non-zero elements of $\operatorname{Coker} \delta^*$ are in bijection to the isomorphism classes of non-trivial infinitesimal deformations of S/I.

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This cokernel is denoted by $T^1(S/I)$ and is called the first cotangent module of S/I.

For any B-algebra homomorphism $B \to A$, there exist functors $T^i(A/B, M)$ and $T_i(A/B, M)$ for $i = 0, 1, \ldots$ the so-called tangent and cotangent functors. They are functor in all three variables.

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In characteristic 0, a different (and simpler approach) is given by Palamadov (Deformations of complex spaces) by using DGA algebras.

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Example: Let $I = (xy, xz, yz) \subset S = K[x, y, z]$, and $L = (xw, xz, yz) \subset T = K[x, y, z, w]$.

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Then $t := w - y$ is a non-zerodivisor of T/L. Thus $K[t] \to T/L$ is flat, and hence $T/L \otimes K[\epsilon]$ with $K[\epsilon] = K[t]/(t^2)$ is an infinitesimal deformation of S/I.

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We have $T = K[x, y, z, t]$ and $L = (xy + xt, xz, yz)$, and hence $T/L \otimes K[\epsilon] \simeq S[\epsilon]/(xy + x\epsilon, xz, yz)$.

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Let $\partial = f\partial_x + g\partial_y + h\partial_z$. Then $x = \partial(xy) = fy + gx$, and hence $f = 0$ and $g = 1$. Furthermore, $0 = \partial(yz) = fz + gx = gx$, and hence $g = 0$, a contradiction.

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The calculations show that $T^1(S/I)_{-1} \neq 0$.

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The module of differentials $\Omega_{R/K}$ is defined by the universal

property that there exists a K-derivation $d : R \to \Omega_{R/K}$ such that for any derivation $\delta : R \to M$ there exists an R-module homomorphism $\varphi : \Omega_{R/K} \to M$ such that

$$\partial = \varphi \circ d.$$

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Thus the relation matrix of $\Omega_{R/K}$ is the Jacobian matrix.

There is the fundamental exact sequence of R-modules

$$I/I^2 \to \bigoplus_{i=1}^{n} R dx_i \to \Omega_{R/K} \to 0,$$

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By dualizing, the fundamental exact sequence yields the exact sequence

$$\delta^* : \bigoplus_{i=1}^{n} R\partial_i \to (I/I^2)^* \to T^1(R) \to 0.$$

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sequence yields

$$\begin{aligned}
\operatorname{Ext}^1_R(\Omega_{R/K}, R) &= \operatorname{Coker}(\bigoplus_{i=1}^n R\partial_i \to U^*) \\
&= \operatorname{Coker}(\bigoplus_{i=1}^n R\partial_i \to (I/I^2)^*) = T^1(R).
\end{aligned}$$

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(i) the ideal I is the image of J under the K-algebra homomorphism $S[y] \to S$ with $y \mapsto x_i$ and $x_j \mapsto x_j$ for all $1 \leq j \leq n$;

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Separations are unobstructed deformations of monomial ideals which preserve the monomial structure.

Proposition. Let I be a squarefree monomial ideal, and let J be an i-separation of I. Then $T^1(S/I)_{-\epsilon_i} \neq 0$.

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Let J be minimally generated by $v_1, \ldots, v_m$. We may assume that y divides $v_1, \ldots, v_k$ but does not divide the other generators of J. We may furthermore assume that for all i, v_i is mapped to u_i under the K-algebra homomorphism (i).

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Then we may write

$$J = (u_1 + (u_1/x_i)(y - x_i), \ldots, u_k + (u_k/x_i)(y - x_i), u_{k+1}, \ldots, u_m).$$

From this presentation and by (iii) it follows that $S[y]/J$ is an unobstructed deformation of S/I induced by the element $[\varphi] \in T^1(S/I)_{-\epsilon_i}$, where $\varphi \in I^*$ is the S-module homomorphism with $\varphi(u_j) = u_j/x_i + I$ for $j = 1, \ldots, k$ and $\varphi(u_j) = 0$, otherwise.

Condition (ii) makes sure that $S[y]/J$ is a non-trivial deformation of S/I.

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Each monomial ideal admits an inseparable model, but in general not only one.

For example, $J = (x_1y, x_1x_3, x_2x_3)$ is an inseparable model of $I = (x_1x_2, x_1x_3, x_2x_3)$.

Problem 1. Let $I = (x_1x_2, x_2x_3, x_3x_4) \subset K[x_1, x_2, x_3, x_4]$. Show that S/I is not rigid.

Problem 2. Let $I \subset S$ be a graded ideal, and assume that K is a perfect field and that $R = S/I$ is a reduced CM ring. Then R is rigid if and only if $\Omega_{R/K} \otimes \omega_R$ is CM.

Problem 3. Let $I \subset S$ be a graded ideal, and assume that K is a perfect field and that $R = S/I$ is a 1-dimensional reduced Gorenstein ring. Then R is rigid if and only if $\Omega_{R/K}$ is torsionfree.

Problem 4. Find an inseparable monomial ideal which is not rigid.