Lecture 4: Bi-Cohen-Macaulay graphs

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August 17-24
Moieciu de Sus, România

Basis properties of bi-CM graphs

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Thus the Eagon-Reiner theorem implies that I is bi-CM if and only if I is a Cohen-Macaulay ideal with linear resolution.

From this description it follows that a bi-CM graph is connected. Indeed, if this is not the case, then there are induced subgraphs $G_1, G_2 \subset G$ such that $V(G)$ is the disjoint union of $V(G_1)$ and $V(G_2)$. It follows that $I(G) = I(G_1) + I(G_2)$, and the ideals $I(G_1)$ and $I(G_2)$ are ideals in a different set of variables.

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Let $C \subset [n]$. Then the monomial prime ideal $P_C = (\{x_i : i \in C\})$ is a minimal prime ideal of $I(G)$ if and only if C is a minimal vertex cover of G. Thus G is unmixed if and only if $I(G)$ is unmixed in the algebraic sense.

A subset $D \subset [n]$ is called an independent set of G if D contains no set $\{i, j\}$ which is an edge of G. Note that D is an independent set of G if and only if $[n] \setminus D$ is a vertex cover. Thus the minimal vertex covers of G correspond to the maximal independent sets of G.

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(d) $\beta_i(I_G) = (i+1)\binom{n-c+1}{i+2}$ for $i = 0, \ldots, n-c-1$.

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(a) $\Leftrightarrow$ (b): We divide $S/I(G)$ by a maximal regular sequence of linear forms to obtain T/J, where J is generated in degree 2 and $\dim T/J = 0$. Now $I(G)$ has a linear resolution if and only if J has a linear resolution, and this is the case if and only if $J = \mathfrak{m}_T^2$. Thus G is bi-CM if and only if the number of generators of J is equal to $\binom{n-c+1}{2}$. Since I_G and J have the same number of generators and since the number of generators of I_G is equal to $|E(G)|$, the assertion follows.

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(b)$\Leftrightarrow$ (c): Since S/I_G is Cohen-Macaulay, the multiplicity of S/I_G is equal to the length $\ell(T/J)$ of T/J. On the other hand, the multiplicity is also the number of minimal prime ideals of I_G which coincides with the number of minimal vertex covers of G. Thus the length of T/J is equal to the number of minimal vertex covers of G. Since $J = \mathfrak{m}_T^2$ if and only if $\ell(T/J) = n - c + 1$, the assertion follows.

(a)$\Rightarrow$ (d): Note that $\beta_i(I_G) = \beta_i(J)$ for all i. Since J is isomorphic to the ideal of 2-minors of the matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-c} & 0 \\ 0 & y_1 & \cdots & y_{n-c-1} & y_{n-c} \end{pmatrix}$$

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(d)⇒ (a): It follows from the description of the Betti numbers of I_G that $\operatorname{proj\,dim} S/I_G = n - c$. Thus, $\operatorname{depth} S/I_G = c$. Since $\dim S/I_G = c$, it follows that I_G is a Cohen-Macaulay ideal. Since $|E(G)| = \beta_0(I_G) = \binom{n-c+1}{2}$, condition (b) is satisfied, and hence G is bi-CM, as desired. □

The classification of bipartite and chordal bi-CM graphs

Theorem. Let G be a bipartite graph on the vertex set V with bipartition $V = V_1 \cup V_2$ where $V_1 = \{v_1, \dots, v_n\}$ and $V_2 = \{w_1, \dots, w_m\}$. Then the following conditions are equivalent:

(a) G is a bi-CM graph;

(b) $n = m$ and $E(G) = \{\{v_i, w_j\}\, 1 \leq i \leq j \leq n\}$.

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The following picture shows a bi-CM bipartite graph for $n = 4$.

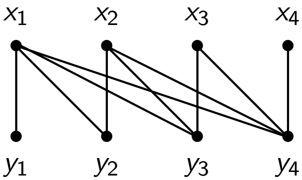

Figure: A bi-CM bipartite graph.

A subset $F \subset [n]$ is called a clique of G, if $\{i, j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$. The set of all cliques of G is a simplicial complex, denoted $\Delta(G)$.

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Theorem.Let G be a chordal graph on the vertex set $[n]$. The following conditions are equivalent:

(a) G is a bi-CM graph;

(b) Let $F_1, \ldots, F_m$ be the facets of the clique complex of G with a free vertex. Then $m = 1$,
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 (i) $V(G) = V(F_1) \cup V(F_2) \cup \ldots \cup V(F_m)$, and this union is disjoint;

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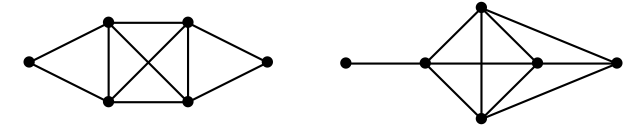
  (iii) the restriction of G to $[n] \setminus \{j_1, \ldots, j_m\}$ is a clique.

The following picture shows, up to isomorphism, all bi-CM chordal graphs whose center is the complete graph K_4 on 4 vertices:



Inseparable graphs

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When is a graph inseparable and what are the separable models of a graph?

Figure: A triangle and one of its inseparable models

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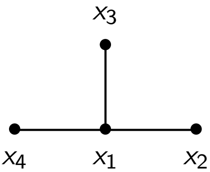
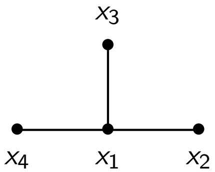
We now define the generic graph G_T associated with T whose vertex set is

$$V(G_T) = \{(i,j),(j,i) : \ \{i,j\} \text{ is an edge of } T\}.$$

and with $\{(i,k),(j,l)\} \in E(G_T)$ if and only if there exists a path P from i to j such that $k = b(i,j)$ and $l = e(i,j)$.

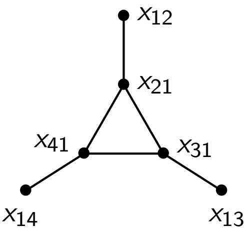


T



T

The generic graph of T.



G_T

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(c) Let G be any Bi-CM graph. Then there exists a tree T such that G_T is an inseparable model of G.

(d) The finitely many trees T for which G_T is an inseparable model of G can all be determined by considering the Alexander dual $I(G)^\vee$ of $I(G)$, and the relation trees of $I(G)^\vee$.

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As we noticed before, the Alexander dual $J = I(G)^\vee$ of the edge ideal of a bi-CM graph G is a Cohen–Macaulay ideal of codimension 2 with linear resolution. The ideal J may have several distinct relation matrices with respect to the unique minimal set of monomial generators of J.

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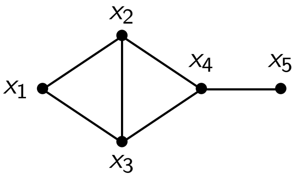
Let $u_1, \ldots, u_{m+1}$ be the unique minimal set of monomial generators of J. Because J has a linear resolution, the generating relations of J may be chosen all of the form $x_k u_i - x_l u_j = 0$. This implies that in each row of the $m \times (m+1)$-relation matrix A there are exactly two non-zero entries (which are variables with different signs). We call such relations, relations of binomial type.

Consider the bi-CM graph G on the vertex set $[5]$ and edges $\{1,2\}$ $\{2,3\}$, $\{3,1\}$, $\{2,4\}$, $\{3,4\}$, $\{4,5\}$.



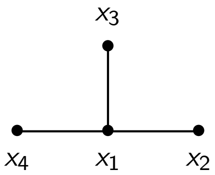
The ideal $J = I_G^\vee$ is generated by $u_1 = x_2x_3x_4$, $u_2 = x_1x_3x_4$, $u_3 = x_2x_3x_5$ and $u_4 = x_1x_2x_4$. The relation matrices with respect to u_1, u_2, u_3 and u_4 are the matrices

$$A_1 = \begin{pmatrix} x_1 & -x_2 & 0 & 0 \\ x_5 & 0 & -x_4 & 0 \\ x_1 & 0 & 0 & -x_3 \end{pmatrix},$$

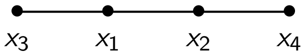
and

$$A_2 = \begin{pmatrix} x_1 & -x_2 & 0 & 0 \\ x_5 & 0 & -x_4 & 0 \\ 0 & x_2 & 0 & -x_3 \end{pmatrix}.$$

In the above example the relation tree of A_1 is



while the relation tree of A_2 is



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$$a_{kl} = \begin{cases} x_{ij}, & \text{if } l = i, \\ -x_{ji}, & \text{if } l = j, \\ 0, & \text{otherwise.} \end{cases}$$

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By the Hilbert-Burch theorem, the matrix A_T is the relation matrix of the ideal J_T of maximal minors of A_T, and J_T is a Cohen-Macaulay ideal of codimension 2 with linear resolution.

Naeem showed: the minors of A_T (which are the generators of J_T) are the monomials

$$\prod_{\substack{i=1 \\ i \neq j}}^{m+1} x_{ib(i,j)} \quad (j = 1, \ldots, m+1),$$

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$$J_T^{\vee} = (x_{ib(i,j)} x_{je(i,j)} : \ 1 \leq i < j \leq m+1).$$

Naeem showed: the minors of A_T (which are the generators of J_T) are the monomials

$$\prod_{\substack{i=1 \\ i \neq j}}^{m+1} x_{ib(i,j)} \quad (j = 1, \ldots, m+1),$$

and

$$J_T^{\vee} = (x_{ib(i,j)} x_{je(i,j)} : \ 1 \leq i < j \leq m+1).$$

Hence $J_T^{\vee} = I(G_T)$ where G_T is the generic graph defined before.

This shows that G_T is a bi-CM graph.

In order to see that G_T is inseparable, we apply the following criterion: let G be a graph on the vertex set $[n]$.

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Theorem. (Altmann, Bigdeli, Dancheng Lu, H) The following conditions are equivalent:

(a) The graph G is inseparable;

(b) $G^{(i)}$ is connected for all i.

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Therefore, for any relation tree T of $I(G)^\vee$, one obtains the inseparable model G_T of G.

Finally one shows that any inseparable bi-CM graph is of the form G_T, and that all inseparable models of G are the graphs G_T with T a relation tree of $I(G)^\vee$.

Problem 1. Which of the ideals $L(P, Q)$ is bi-CM?

Problem 2. Which of the polymatroidal ideals are bi-CM?

Problem 3. Which of the matroidal ideals are inseparable?