Lecture 5: Rigidity and separability of simplicial complexes and toric rings

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From Lecture 3 we know that $K[\Delta]$ is rigid if and only if $T^1(K[\Delta]) = 0$.

Since $T^1(K[\Delta])$ is $\mathbb{Z}^n$-graded, it follows that $T^1(K[\Delta]) = 0$ if and only if $T^1(K[\Delta])_{\mathbf{c}} = 0$ for all $\mathbf{c} \in \mathbb{Z}^n$.

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(b) Assuming $\mathbf{b} \in \{0,1\}^n$, then $T^1(\Delta)_{\mathbf{a}-\mathbf{b}}$ depends only on A and B.

Recall that for a subset A of $[n]$, the link of A is defined to be

$$\mathrm{link}_\Delta A = \{F \in \Delta \quad F \cap A = \emptyset, \ F \cup A \in \Delta\}$$

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We say that Δ is $\emptyset$-rigid, if $T^1(\Delta)_{-\mathbf{b}} = 0$ for all $\mathbf{b} \in \{0,1\}^n$.
Thus, Δ is rigid, if and only if all its links are $\emptyset$-rigid.

Let Δ_1 and Δ_2 be simplicial complexes on disjoint vertex sets, then the join $\Delta_1 * \Delta_2$ is a simplicial complex on the vertex set $V(\Delta_1) \cup V(\Delta_2)$ with faces $\{F \cup G : \; F \in \Delta_1, G \in \Delta_2\}$.

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Theorem. (Altmann, Bigdeli, H, Danchen Lu) (a) Let $I_{\Delta_1} \subseteq K[x_1, \dots, x_n]$ and $I_{\Delta_2} \subseteq K[y_1, \dots, y_m]$. Then

$$T^1(\Delta_1 * \Delta_2) = T^1(\Delta_1)[y_1, \dots, y_m] \oplus T^1(\Delta_2)[x_1, \dots, x_n].$$

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(b) Let $\Delta_1 \neq \{\emptyset\}$ and $\Delta_2 \neq \{\emptyset\}$ be simplicial complexes with disjoint vertex sets, and asssume that for $i = 1, 2$, $\{j\} \in \Delta_i$ for all $j \in V(\Delta_i)$. Then the following conditions are equivalent:

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(3) Δ_1 and Δ_2 are simplices with $\dim \Delta_1 + \dim \Delta_2 > 0$.

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For $i \in G$ we defined in Lecture 4, the neighborhood $N(i) = \{j : \{i,j\} \in E(G)\}$, and denoted by $G^{(i)}$ the complementary graph of the restriction $G_{N(i)}$ of G to $N(i)$.

We also define the sets

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We have the following criterion of rigidity of G.

Theorem. G is rigid if and only if for all independent sets $A \subseteq V(G)$ one has:

(α) $(G \setminus N[A])^{(i)}$ is connected for all $i \in [n] \setminus N[A]$;

(β) $G \setminus N[A]$ contains no isolated edge.

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Proof. If G is not connected, then $\Delta(G)$ is a join. Thus G is rigid (resp. CM) if and only if each connected component is rigid (resp. CM). Thus we assume that G is connected. Since G is Cohen–Macaulay, after a suitable relabeling of its vertices, G arises from a finite poset $P = \{p_1, \ldots, p_n\}$ as follows:
$V(G) = \{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ and $E(G) = \{\{p_i, q_j\} \;\; p_i \leq p_j\}$.
We may assume that p_1 is a minimal element in P. Let $A = \{p_2, \ldots, p_n\}$. Then $N[A] = \{p_2, \ldots, p_n, q_2, \ldots, q_n\}$, and $G \setminus N[A] = \{p_1, q_1\}$. It follows from (β) that G is not rigid. $\square$

A vertex v of G is called a free vertex if $\deg v = 1$, and an edge e is called a leaf if it has a free vertex. An edge e of G is called a branch, if there exists a leaf e' with $e' \neq e$ such that $e \cap e' \neq \emptyset$.

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Theorem. (Altmann, Bigdeli, H, Dancheng Lu) Let G be a graph on the vertex set $[n]$ such that G does not contain any induced cycle of length $4, 5$ or 6. Then G is rigid if and only if each edge of G is a branch and each vertex of a 3-cycle of G belongs to a leaf.

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Corollary. Let G be a chordal graph. Then G is rigid if and only if each edge of G is a branch and each vertex of a 3-cycle of G belongs to a leaf.

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Corollary. Let G be a chordal graph. Then G is rigid if and only if each edge of G is a branch and each vertex of a 3-cycle of G belongs to a leaf.

Corollary. Suppose that all cycles of G have length ≥ 7 (which for example is the case when G is a forest). Then G is rigid if and only if each edge of G is a branch.

T^1 for toric rings

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The toric ring $K[H]$ associated with H is the K-subalgebra of the ring $K[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ of Laurent polynomials generated by the monomials $t^{h_1}, \dots, t^{h_n}$. Here $t^a = t_1^{a(1)} \cdots t_m^{a(m)}$ for $a = (a(1), \dots, a(m)) \in \mathbb{Z}^m$.

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Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over K in the variables $x_1, \dots, x_n$. The K-algebra $R = K[H]$ has a presentation $S \to R$ with $x_i \mapsto t^{h_i}$ for $i = 1, \dots, n$.

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The kernel $I_H \subset S$ of this map is the toric ideal attached to H. Corresponding to this presentation of $K[H]$ there is a presentation $\mathbb{N}^n \to H$ of H which can be extended to the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^m$ with $\epsilon_i \mapsto h_i$ for $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ denotes the canonical basis of $\mathbb{Z}^n$.

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We define an H-grading on S by setting $\deg x_i = h_i$. Then I_H is a graded ideal with $\deg f_v = h(v)$, where

$$h(v) = \sum_{i,\ v(i) \geq 0} v(i) h_i \ (= \sum_{i,\ v(i) \leq 0} -v(i) h_i).$$

Let $v_1, \ldots, v_r$ be a basis of L. Since I_H is a prime ideal we may localize S with respect to this prime ideal and obtain

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The cotangent module $T(K[H])$ admits a natural $\mathbb{Z}H$-grading.

The module of differentials has a presentation

$$\Omega_{R/K} = (\bigoplus_{i=1}^{n} R dx_i)/U,$$

where U is the submodule of the free R-module $\bigoplus_{i=1}^{n} R dx_i$ generated by the elements df_v with $v \in L$, where

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There is an epimorphism $\chi : \Omega_{R/K} \to \mathfrak{m}$ with $\chi(dx_i) \mapsto h_i t^{h_i}$ where $\mathfrak{m} = (t^{h_1}, \dots, t^{h_n})$ is the graded maximal ideal of R.

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Since $\operatorname{rank} \Omega_{R/K} = \operatorname{rank} \mathfrak{m} = 1$, it follows that $C = \operatorname{Ker} \chi$ is a torsion module. Thus we obtain the following exact sequence

$$0 \to C \to \Omega_{R/K} \to \mathfrak{m} \to 0,$$

which induces the long exact sequence

$$\operatorname{Hom}_R(C, R) \to \operatorname{Ext}^1_R(\mathfrak{m}, R) \to \operatorname{Ext}^1_R(\Omega_{R/K}, R).$$

Since R is a 1-dimensional domain, R is Cohen-Macaulay. Thus $\mathrm{Hom}_R(C, R) = 0$ and $\mathrm{Ext}^1_R(\mathfrak{m}, R) \simeq \mathfrak{m}^{-1}/R \neq 0$. It follows that $\mathrm{Ext}^1_R(\Omega_{R/K}, R) \neq 0$.

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It is a big open conjecture whether a K-subalgebra $R \subset K[t]$ is rigid if and only if R is regular.

The conjecture is known to be correct if the embedding dimension of R is 3, or R is Gorenstein of embedding dimension 4. The proof uses Hilbert-Burch and the Buchsbaum-Eisenbud structure theorem.

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Let $a \in \mathbb{Z}H$. We denote by KL the K-subspace of K^n spanned by $v_1, \ldots, v_s$ and by KL_a the K-subspace of KL spanned by the set of vectors $\{v_i : \ a + h(v_i) \notin H\}$.

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Then one shows that $\dim_K (U^*)_a = \dim_K KL - \dim_K KL_a$. for all $a \in \mathbb{Z}H$.

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In conclusion one sees that all information which is needed to compute $\dim_K T^1(R)_a$ can be obtained from the $(s \times n)$-matrix

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Indeed, $\dim_K T^1(K[H])_a$ can be computed as follows: let $l = \operatorname{rank} A_H$, l_a the rank of the submatrix of A_H whose rows are the ith rows of A_H for which $a + h(v_i) \notin H$, and let d_a be the rank of the submatrix of A_H whose columns are the jth columns of A_H for which $a + h_j \in H$.

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Corollary. $T^1(K[H]))_a = 0$ for all $a \in H$.

Separated saturated lattices

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This trivial case we do not consider as a proper solution of finding an $K[H']$ that specializes to $K[H]$. If no non-trivial $K[H']$ exists, which specializes to $K[H]$, then H will be called inseparable and otherwise separable.

Let $\epsilon_1, \ldots, \epsilon_n$ be the canonical basis of $\mathbb{Z}^n$ and $\epsilon_1, \ldots, \epsilon_n, \epsilon_{n+1}$ the canonical basis of $\mathbb{Z}^{n+1}$. Let $i \in [n]$. We denote by $\pi_i : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ the group homomorphism with $\pi_i(\epsilon_j) = \epsilon_j$ for $j = 1, \ldots, n$ and $\pi_i(\epsilon_{n+1}) = \epsilon_i$.

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For convenience we denote again by π_i the K-algebra homomorphism $S[x_{n+1}] \to S$ with $\pi_i(x_j) = x_j$ for $j = 1, \dots, n$ and $\pi_i(x_{n+1}) = x_i$.

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Let $L \subset \mathbb{Z}^n$ be a saturated lattice. We say that L is *i-separable* for some $i \in [n]$, if there exists a saturated lattice $L' \subset \mathbb{Z}^{n+1}$ such that

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(ii) $\pi_i(I_{L'}) = I_L$;

(iii) there exists a minimal system of generators $f_{w_1}, \dots, f_{w_s}$ of $I_{L'}$ such that the vectors $(w_1(n+1), \dots, w_s(n+1))$ and $(w_1(i), \dots, w_s(i))$ are linearly independent.

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If L' is an i-separation lattice of L, then $x_{n+1} - x_i$ is a non-zerodivisor on $S[x_{n+1}]/I_{L'}$ and

$$(S[x_{n+1}]/I_{L'})/(x_{n+1} - x_i)(S[x_{n+1}]/I_{L'}) \simeq S/I_L.$$

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Theorem. Let H be a positive affine semigroup which is minimally generated by $h_1, \ldots, h_n$, $L \subset \mathbb{Z}^n$ the relation lattice of H. Suppose that L is i-separable. Then $T^1(K[H])_{-h_i} \neq 0$. In particular, if $K[H]$ is standard graded, then H is inseparable, if $T^1(K[H])_{-1} = 0$.

Proposition. Any numerical semigroup ring $K[t^{h_1}, t^{h_2}, t^{h_3}]$ is i-separable for $i = 1, 2, 3$.

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Theorem. (Bigdeli, H, Dancheng Lu) Let G be a bipartite graph with edge set $\{e_1, \ldots, e_n\}$, and let $R = K[G]$ be the edge ring of G. Then the following conditions are equivalent:

(a) The relation lattice of $H(G)$ is i-separable.

(b) $T^1(R)_{-h_i} \neq 0$.

(c) There exists a cycle C of G for which e_i is a chord, and there is no crossing path chord P of C with respect to e_i.

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It is widely open for which graphs G, the edge ring $K[G]$ is rigid.

Problem 1. Let $\mathfrak{m}$ be the graded maximal ideal of $S = K[x_1, \dots, x_n]$. Compute the module $T^1(S/\mathfrak{m}^2)$.

Problem 2. Let $I \subset \mathfrak{m}^2$ be a graded ideal with $\dim S/I = 0$. Do we always have that $T^1(R) \neq 0$?

Problem 3. Let $R = K[H]$ be a numerical semigroup ring. Show that $T^1(R)$ is module of finite length.

Problem 4. Compute the length of $T^1(R)$ when $R = K[t^{h_1}, t^{h_2}]$.