Lefschetz properties for balanced 3-polytopes

Martina Juhnke-Kubitzke
(joint work with David Cook II, Satoshi Murai and Eran Nevo)

Institute of Mathematics, University of Osnabrück

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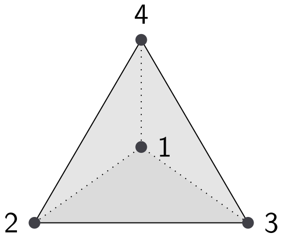
A simplicial d-dimensional polytope is balanced, if its boundary complex is balanced.

The boundary of the d-simplex

For $d \in \mathbb{N}$ let $\partial\Delta_d = \{F \ : \ F \subsetneq \{1, 2, \ldots, d+1\}\}$ be the boundary of the d-simplex Δ_d.



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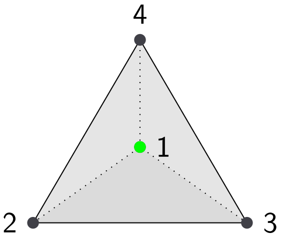

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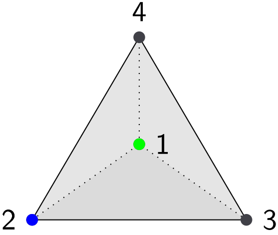

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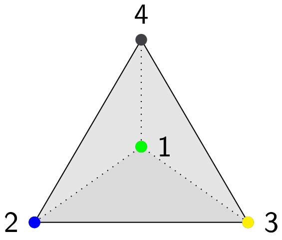

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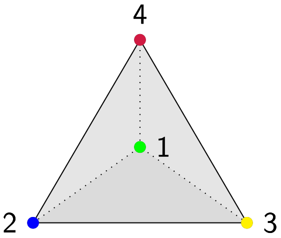

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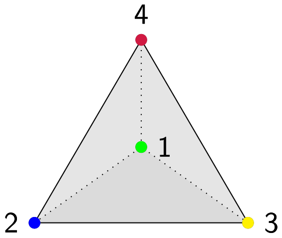

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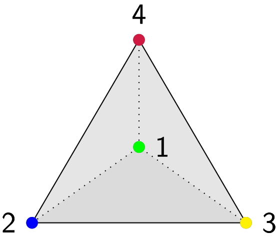
$\Rightarrow$ $\partial\Delta_d$ is not balanced.



The d-dimensional cross-polytope

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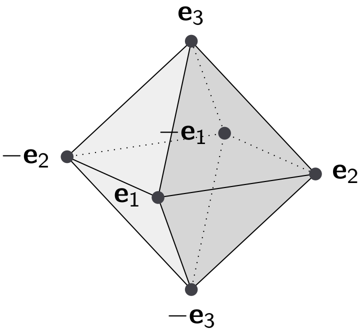

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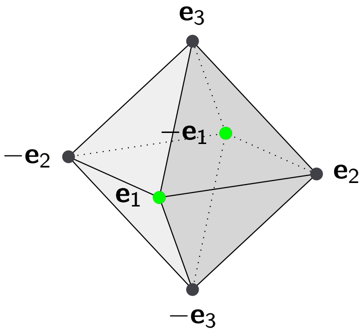

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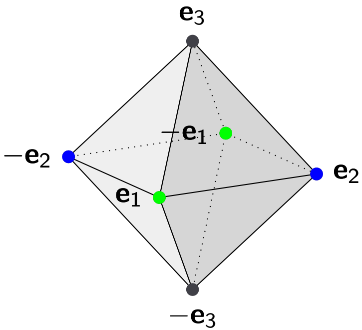

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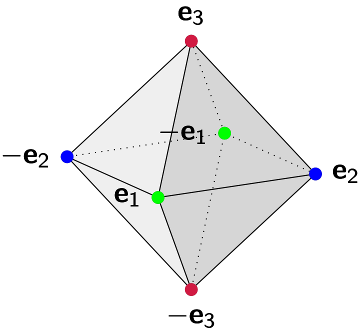

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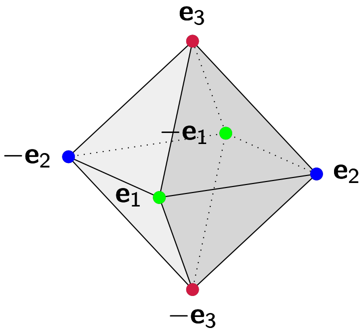
defines a coloring of $\partial\mathcal{C}_d$ with d colors.



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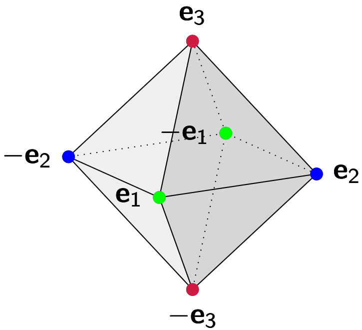

$\Rightarrow$ $\mathcal{C}_d$ is balanced.

Lefschetz properties

$\mathbb{F}$ infinite field

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A has the strong Lefschetz property if there exists a linear form $\omega \in A_1$ such that the multiplication map

$$\times\omega^{s-2\ell} : \; A_\ell \to A_{s-\ell}$$
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Lefschetz properties are a tool to obtain information/conditions on the Hilbert function of A.

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Question:

Does $\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]$ have the strong Lefschetz property if Θ is not generic?

Back to balanced simplicial polytopes

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Theorem (Cook II, J.-K., Murai, Nevo)

Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2, 3$.

Let Δ be the boundary complex of a simplicial d-polytope, and

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Then $\mathbb{F}[\Delta]/\Theta_c\mathbb{F}[\Delta]$ *has the strong Lefschetz property.*

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Note:

If $\mathrm{char}(\mathbb{F}) \in \{2, 3\}$, then $\omega^3 = 0$ for any linear form $\omega \in \mathbb{F}[\Delta]/\Theta^{(c)}\mathbb{F}[\Delta]$.

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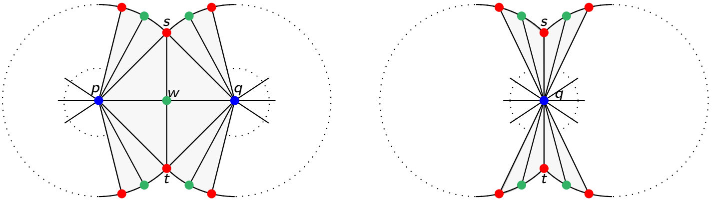

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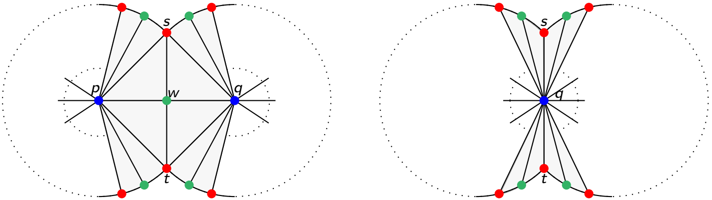

Verify the claim for $\mathcal{C}_3$ and show that balanced connected sum and inverse balanced contraction preserve the colored strong Lefschetz property. □

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such that for any facet two of its vertices are colored blue and one vertex is colored red.

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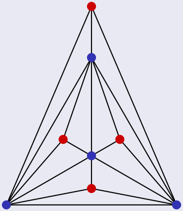
Example

Let Δ be the boundary of the simplicial 3-polytope obtained by subdividing each facet of the 3-simplex in 3 triangles by adding a new (red) vertex.



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Stanley showed, that if Δ is $(2,1)$-balanced, there exists a $(2,1)$-colored linear system of parameters $\Theta = \{\theta_1, \theta_2, \theta_3\}$ for $\mathbb{F}[\Delta]$ such that

- θ_1 and θ_2 are linear combinations of blue vertices, and
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Question

Is there an analogous result as for balanced simplicial polytopes?

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(2) *For any subset $W \subseteq U$ with $|W| \geq 2$, the induced subcomplex*

$$\Delta_W = \{F \in \Delta \ : \ F \subseteq W\}$$

has at most $2|W| - 3$ edges.

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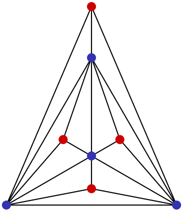

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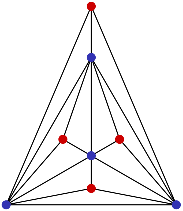

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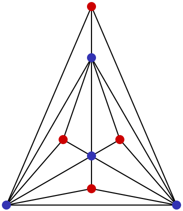

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many edges. Hence, Δ does not have the strong Lefschetz property w.r.t. a $(2, 1)$-colored linear system of parameters.

Thank you for your attention!