II Decompositions and Connectivity

Yesterday: Analyze connectivity of walks on lattice points in $\mathbb{N}^n$ using binomial algebra via

Prop: $u \sim_{\mathcal{M}} v \iff x^u - x^v \in I_{\mathcal{M}} = \langle x^{m^+} - x^{m^-} : m \in \mathcal{M} \rangle$

First idea: Understand the graph on $\mathbb{N}^n$ by ~~decompo~~ decomposition of $I_{\mathcal{M}}$ (Diaconis - Eisenbud - Sturmfels)

$$I_{\mathcal{M}} = I_A \cap I_1 \cap \dots \cap I_s$$

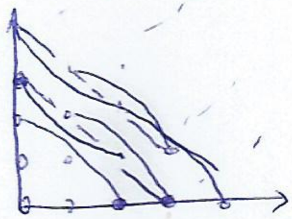
This gives necessary conditions for connectivity that are jointly sufficient.

For example if $\mathcal{M}$ spans $\ker_{\mathbb{Z}} A$ then I_A is a minimal prime of $I_{\mathcal{M}}$

$I_A = I_{\mathcal{M}} : \left(\prod_{i=1}^{n} x_i\right)^{\infty}$ Since I_A is prime, it is an associated prime.

Examples: $A = (1, 1)$ (standard grading)

$\mathcal{M} = \left\{ \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \end{pmatrix} \right\}$ $I_{\mathcal{M}} = \langle x^2 - y^2, x^3 - y^3 \rangle$



Decompose $I_{\mathcal{M}}$:

$I_A = \langle x - y \rangle$

Useful Lemma: R comm. ring $I \subseteq R$ ideal, $f \in R$, $n \in \mathbb{N}$

Thm $I = (I : f^u) \cap (I + f^u)$

Pf: $I \subseteq$ r.h.s. ✓ Let $g \in$ r.h.s $\Rightarrow g = \overset{I}{a} + \overset{R}{b} \cdot f^u$
and $g \cdot f^u \in I \Rightarrow b \cdot f^{2u} \in I \overset{(Hyp)}{\Rightarrow} b \cdot f^u \in I$
$\Rightarrow g \in I$. □

<u>Side remark:</u> Could turn this algorithmic
Since if I is not primary then $\exists f$ s.t.
$I : f^u \neq I$ and $I : f^u \neq R$ and allows
going closer to a primary decomposition.

<u>Back to example:</u> $I_u = \langle x^2 - y^2, x^3 - y^3 \rangle$

$x^2(x-y) \in I_u$

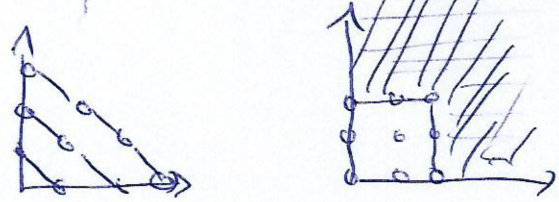
$(x-y) \in I_u : x^2 \rightarrow I_u : x^2 = I_A$

By Lemma: $I_u = I_A \cap \langle x^2, y^2 \rangle$ $\quad x^3y$

two necc. cond. for $u \sim_u v$

(1) $Au = Av$

(2) $\max\{u_1, u_2\} \geq 2$
$\max\{v_1, v_2\} \geq 2$



<u>Observe:</u> Intersection of two graphs equals graph of I_u
The idea to use this for connectivity analysis is
implemented e.g. in
"Positive margins and primary decomposition" (KRS)

Want to talk about how to analyze decompositions of binomial ideals using combinatorics (pictures, congruences on monoids)

<u>Based on</u>: "Decompositions of comm. monoid congruences and binomial ideals" (KM)

- The <u>vertex set</u> of the above graphs is a <u>monoid</u> (comm. fin. generated) (Noetherian)
- Edges form a congruence, an <u>equivalence relation</u> on the monoid Q that satisfies

$$u \sim v \Rightarrow a+u \sim a+v \quad \forall a,u,v \in Q$$

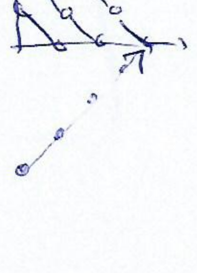
Examples so far: · $Q = \mathbb{N}^n$

<u>Congruences from ideals</u> $I \subseteq \Bbbk[\mathbb{N}^n]$

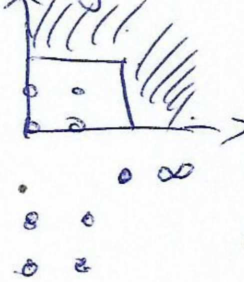
$$u \sim_I v \Leftrightarrow x^u - \lambda x^v \in I \text{ some } \lambda \neq 0, \lambda \in \Bbbk$$

<u>Exercise</u>: Generalize this to arbitrary rings

The quotient $Q/\sim$ is a monoid again



$$\Bbbk[x,y]/\langle x-y\rangle \cong \Bbbk[x]$$



∞

<u>Def</u>: An element $\infty \in Q$ is <u>nil</u> if $a + \infty = \infty \quad \forall a \in$

$$\Bbbk[x,y]/\langle x^2,y^2\rangle = \Bbbk[\mathbb{N}^n/\sim]/\langle x^\infty\rangle$$

These monoids finely grade the quotient rings

Noeth. monoid Q iff I is a binomial ideal.

ES Def: I generated by polys of the form $x^u - \lambda x^v$ $\lambda \in k$

$\mathcal{M}$ spans a saturated lattice

if $(\mathrm{span}_{\mathbb{Q}} \mathcal{M} \cap \mathbb{Z}^n) = \mathrm{span}_{\mathbb{Z}} \mathcal{M}$

$\Leftrightarrow \mathbb{Z}^n / \mathrm{span}\, \mathcal{M}$ has no torsion.

$(2) \subseteq \mathbb{Z}$