III Irreducible decomposition

Def: An ideal $I \leq R$ ($\leftarrow$ comm. Noeth.) is <u>irreducible</u> if:

$$I = I_1 \cap I_2 \quad \text{then} \quad I = I_1 \text{ or } I = I_2$$

Irreducible decomp: $I = I_1 \cap \dots \cap I_s$
I_j irreducible

irredundant if omitting one of the I_j changes intersection.

You may have heard about it in the context of <u>primary decomposition</u>

$$I = Q_1 \cap \dots \cap Q_s \quad \text{such that}$$

Q_j is primary $\Leftrightarrow R/Q_j$ $\ni f$ is regular or nilpotent.

These dec. are geometric since the radicals of the Q_i in an irredundant primary dec. are exactly the associated primes $\mathrm{Ass}(R/I)$

$\Rightarrow$ Decomposition of $\mathrm{Spec}(R/I)$

Common proofs of primary dec:

1) Irreducible dec exists.
2) Irreducible ideals are primary
3) Optional: Gather irred. components with the same radical (= ass. prime)

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In particular if the socle is simple ⟺ 1-dim'l then I is irreducible.

Pf. W. Vasconcelos: "Computational methods...."

Remarks: • R/I is a finite-dim'l R/p vector space.

In 1921 Emmy Noether showed that the number of irreducible components in an irredundant irr. decomposition is an invariant of the ideal.

In 1934 Gröbner characterized this number. ~~...~~ His main tool: If I is P-primary, then $P^n \subseteq I$ for some n large enough. This means the localized quotient R_P/I_P is of finite length over R_P/P_P.

<u>Prop.</u> Let (R,P) be Noetherian local and $I \subseteq R$ a P-primary ideal. Let

$$\mathrm{soc}_P(I) = \{ f \in R/I : P \cdot f = 0 \}$$

be the <u>socle</u> of I (at P).

Any irredundant irreducible dec. of I has

$$\dim_{R/P} \mathrm{soc}_P(I) \quad \text{components.}$$

In particular if the socle is simple $\Leftrightarrow$ 1-dim'l then I is irreducible.

<u>Pf.</u> W. Vasconcelos: "Computational methods...."

<u>Remarks:</u> • R/I is a finite-dim'l R/P vector space.

- Gröbner analysed the chain of ideals in R/I

$$\bar{0} \subseteq \bar{0}:\bar{P} \subseteq \bar{0}:\bar{P}^2 \subseteq \dots \subseteq \bar{0}:\bar{P}^n = R/I$$

The socle is the first layer in this chain.

- socle is an architectural term. A column rests on its socle.

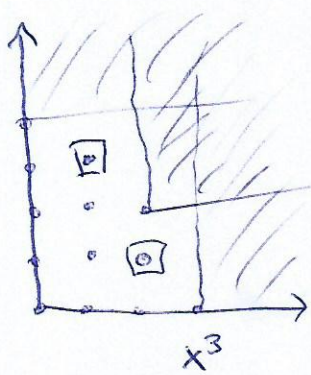
Primary decomposition lifts this to the general case (non-local)

<u>Prop.</u> If $I = I_1 \cap \dots \cap I_s$ is irredundant irr. dec.
then $\{\sqrt{I_j},\ j = 1,\dots,s\} = \mathrm{Ass}(R/I)$

If $P \in \mathrm{Ass}(R/I)$ and $\mathrm{soc}_P(I)$ is the submodule of R/I whose elements are annihilated by P.
Then the number of P-primary irreducible components is the R_P/P_P - dimension of $\mathrm{soc}_P(I_P)$.

<u>Example:</u> <u>Monomial ideals</u> Every monomial ideal has an irreducible decomposition into monomial ideals.

<u>Eg.</u> $I = \langle x^3, x^2y^2, y^4 \rangle \subseteq k[x,y]$

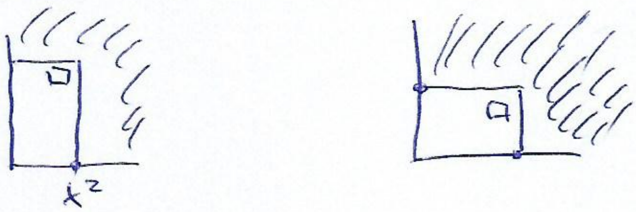


R/I is local with maximal ideal $\langle x, y \rangle = P$
I is P-primary (0 is P-primary in R/I)

$soc_p(I) = \mathbb{k}\{x^2y, xy^3\}$ the outer corners of R/I

Irreducible dec:

$$I = \langle y^4, x^2\rangle \cap \langle y^2, x^3\rangle$$



Note: The same example but with $I \subseteq \mathbb{k}[x,y,z]$

Nothing changes really except that the socle is a vector space over $\mathbb{k}[x,y,z]_{\langle x,y\rangle} / \langle x,y\rangle_{\langle x,y\rangle} \cong \mathbb{k}(z)$

Next goal: Binomial ideals:

In "Binomial ideals" (Eisenbud, Sturmfels '96) it is shown that if $\mathbb{k}$ is algebraically closed, every binomial ideal in $\mathbb{k}[x_1 \ldots x_n]$ has a primary dec. into binomial ideals.

Compare: $\langle x^3 - 1\rangle = \langle x-1\rangle \cap \langle x^2+x+1\rangle$ in $\mathbb{Q}[x]$

Question: In this situation, does every binomial ideal have an irreducible dec. into binomial ideals?

KMO: The answer is no. But, irreducible dec. of binomial ideals is combinatorial.