Lectures 4+5: Irreducible decomposition of binomial ideals

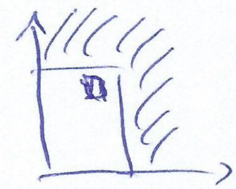
<u>Thm (KM)</u> Every binomial ideal $I \subseteq \mathbb{k}[x_1, \dots, x_n] = \mathbb{k}[\mathbb{N}^n]$

can be expressed as

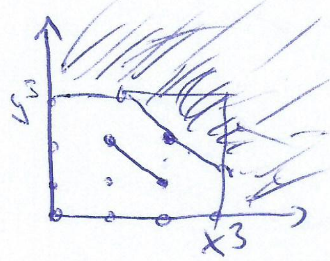
$$I = \bigcap_i J_i \qquad \text{such that}$$

- J_i are binomial, (meso) primary and coprincipal.

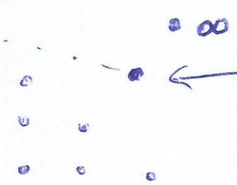
<u>Remarks</u>:
- just primary requires $k = \bar{k}$
- coprincipal generalizes the "one outer corner" condition of irreducible monomial ideals



<u>Ex</u>: $\langle xy^2 - x^2y, x^3, y^3 \rangle = I$



Quotient $\mathbb{N}^n/\sim_I$



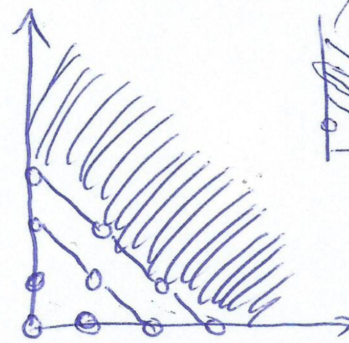
$\leftarrow$ cogenerator modulo $\sim_I$

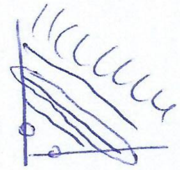
$I \subseteq \mathbb{k}[x, y, z]$ there is an entire class of monomials that cogenerate the component. But the are z-translates of a unique monomial.

<u>Think</u>: Coprincipal $\Leftrightarrow$ unique monomial in the socle up to invertible directions.

Coprincipal is <u>not</u> a simple socle condition since there can be binomials or larger polys in the socle.

First we deal with binomials in the socle.



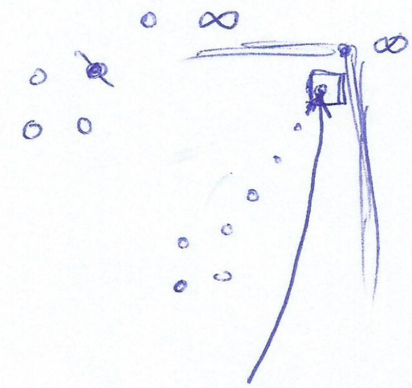


Ex: $\langle x^2-xy,\ xy-y^2,\ x^3\rangle$

is $\langle x,y\rangle$-coprincipal, but

$x-y \in \mathrm{soc}(I)$

can be seen combinatorically



Def: $\sim$ congruence on Q (e.g. $\mathbb{N}^n$)

$P \subseteq Q$ a monoid prime ideal.

Localization Q_P make every element outside of P invertible $\Rightarrow$ make monomials outside of $m_P = \langle x_i : e_i \in P\rangle$ invertible.

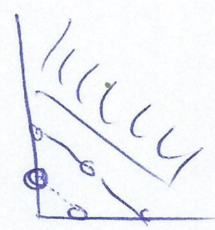
Cogenerator x^4

$x \cdot x^4 \in$

- $w \in Q$ is a key witness for P if there exists an $a \in Q$ such that for all $p \in P$

$\bar{w} \neq \bar{a}$ but $\bar{w}+\bar{p} = \bar{a}+\bar{p}$

and in $\{\bar{w}, \bar{a}\}$ $\bar{w}$ is not maximal.



In the example x, y are witnesses, and so are x^2, xy, y^2 because $\overline{x^2} \neq \overline{\infty}$ but $\overline{x^3} = \overline{\infty}$, $\overline{x^2y} = \overline{\infty}$

Def: A congruence is ~~socular~~ soccular if its key witnesses generate the same principal ideal in Q_P

Def. The soccular collapse of a congruence identifies a and b if a and b are not cogenerators but become equal upon addition of any $p \in P$.

After doing this finitely many times, the result is a soccular congruence, the soccular closure of ~~the~~ $\sim$.

Lift this to binomial ideals:

Def: Fix $I \subseteq \Bbbk[\mathbb{N}^n]$ a binomial ideal and P a monoid prime. (m_P a monomial prime)
The P-socle of I is

$$\mathrm{soc}_P(I) = \left\{ f \in \Bbbk[\mathbb{N}^n]_P / I_P \; : \; m_P \cdot f = 0 \right\}$$

Here $-_P$ means monomial localization (i.e. inverting monomials outside m_P)

Def. A binomial is binoccular if it is P-coprincipal and every binomial in its P-socle is a (monomial) cogenerator. ($\mathrm{char}(k) \neq 2$ here)

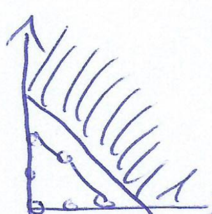
But: Arithmetics problems. $I = \langle x^2 - xy, xy + y^2 \rangle$

More $\cong \mathbb{F}_3^3$

$(x \pm y) \notin \mathrm{soc}$

because $x^2 + xy \notin I$

and $xy - y^2 \notin I$

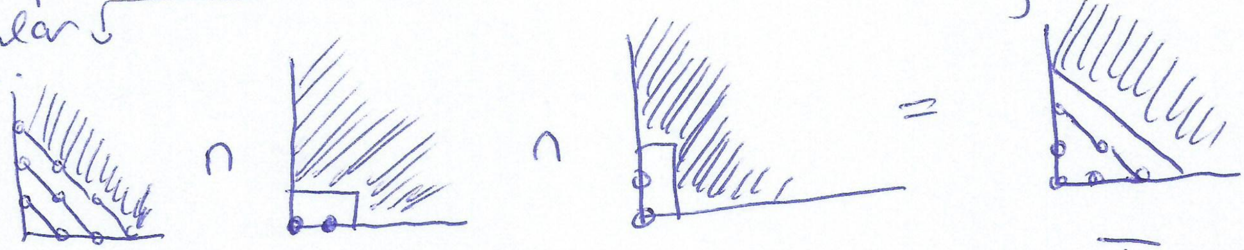


From here define binoccular closure, then show that the intersection of binoccular closures of coprincipal components for key witnesses yields the original ideal.

Ex: $\langle x^2-xy, xy-y^2, x^3 \rangle = I$ Witnesses:

- x^2-class
- x
- y

1) Make coprincipal at witness
2) Binoccular closure



$\langle x-y, x^3 \rangle \cap \langle x^2, y \rangle \cap \langle x y^2 \rangle = I$

Lemma: Fix $I \subseteq \mathbb{K}[\mathbb{N}^n]$ binomial ideal and $W_1, \dots, W_s$ arbitrary ideals containing I. The following are equivalent:

1) $I = W_1 \cap \dots \cap W_s$
2) The natural map $R/I \to \bigoplus_{i=1}^{s} R/W_i$ is injective.
3) The natural map $\mathrm{soc}_P(I) \to \bigoplus_i R_P/(W_i)_P$ is injective for any monoid prime P associated to I.
4) The nat. map. $\mathrm{soc}_{\mathfrak{p}} \to \bigoplus_i R_{\mathfrak{p}}/(W_i)_{\mathfrak{p}}$ is injective for any $\mathfrak{p} \in \mathrm{Ass}(R/I)$.

Pf: (1) $\Leftrightarrow$ (2): The kernel of $R \to \bigoplus_i R/W_i$

Assume (1) and (2).

Localization is exact:

$$R_{\mathfrak{p}}/I_{\mathfrak{p}} \to \oplus R_{\mathfrak{p}}/(W_i)_{\mathfrak{p}} \quad \text{is injective.}$$

$$\mathrm{soc}_{\mathfrak{p}}(I) \subseteq R_{\mathfrak{p}}/I_{\mathfrak{p}} \quad \text{submodule} \Rightarrow (3)$$

(3) $\Rightarrow$ (4) Localization stile exact.

$\mathfrak{p}$ be associated to I. gives by ES an associated monoid prime P (resp. m_P)

with $m_P \subseteq \mathfrak{p}$ but every monomial outside of $\mathfrak{p}$ lies outside of m_P

$\Rightarrow$ Localization at $\mathfrak{p}$ inverts more than in (3)

$$(\mathrm{soc}_P(I))_{\mathfrak{p}} \to \bigoplus_i R_{\mathfrak{p}}/(W_i)_{\mathfrak{p}} \quad \text{injective}$$

but $\mathrm{soc}_{\mathfrak{p}}(I) \subseteq (\mathrm{soc}_P(I))_{\mathfrak{p}}$. $\Rightarrow$ (4).

(4) $\Rightarrow$ (1). Fix $f \in R/I$. Let $\mathfrak{p}$ be minimal prime over the annihilator of f.

Then $\bar{f} \in R_{\mathfrak{p}}/I_{\mathfrak{p}}$ is not zero.

Since $\mathfrak{p}$ is minimal some power of $\mathfrak{p}$ annihilates f, or: $\exists a \in \mathfrak{p}$. $\bar{a} \cdot \bar{f} \neq 0$ but annihilated by $\mathfrak{p}$

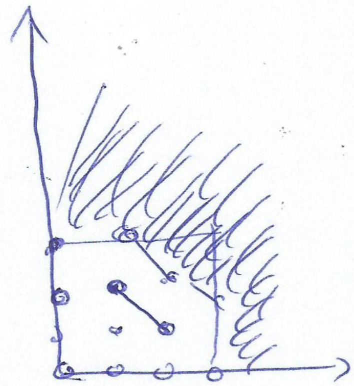
$\bar{a}\bar{f} \in \mathrm{soc}_{\mathfrak{p}}(I)$ so $\bar{a}\bar{f} \neq 0$ mod one $(W_{i^*})_{\mathfrak{p}}$ by (4)

but then $af \notin W_{i^*} \Rightarrow f \notin W_{i^*}$

□

Nonexistence of binomial irred. dec.

Ex: $I = \langle x^2y - xy^2, x^3, y^3 \rangle$



This is coprincipal, binoccular.

$\langle x, y \rangle$ - primary.

cogenerator is $x^2y \sim y^2x$

$$\operatorname{soc}(I) = \mathbb{k}\{\underbrace{x^2+y^2-xy}_{\alpha}, \underbrace{x^2y}_{\beta}\}$$

Any irreducible dec. has two components.

$I = I_1 \cap I_2$ By Lemma $R = \mathbb{k}[x,y]$

$$R/I \to R/I_1 \oplus R/I_2 \quad \text{is inj.}$$

$$\operatorname{soc}(I) \to \operatorname{soc}(I_1) \oplus \operatorname{soc}(I_2) \quad \text{is injective.}$$

Iso of 2d VS over $\mathbb{k}$. Let $f = \alpha + \lambda\beta$ be a generator of $\operatorname{soc}(I_1)$ for some λ.

for dimension reasons $f \in I_2$. Now remains to show. $(I+f) = I_2$, have $I + f \subseteq I_2$ and thus a map $\operatorname{soc}(I+f) \to \operatorname{soc}(I_2)$ from $R/(I+f) \to R/I_2$.

Since $R/I \to R/I+f \to R/I_2$

$$\underset{2d}{\operatorname{soc}(I)} \to \underset{1d}{\operatorname{soc}(I+f)} \to \underset{1d}{\operatorname{soc}(I_2)}$$

$\operatorname{soc}(I+f)$ spanned by α and β but modulo f

$(I+f)$ is irreducible and equals I_2

$\rightarrow$ I_2 is not binomial. $\square$

Problem: Which binomial have binomial irreducible decompositions?

To produce irreducible dec, assume for simplicity Artinian quotient R/I coprincipal with cogenerator x^ω

Decompose R/I as $\Bbbk\{x^\omega\} \oplus \omega^\perp$

Let $\omega_\infty^\perp \subseteq \omega^\perp$ be the largest R/I submodule.

$$Irr(I) = \text{"} I + \omega_\infty^\perp \text{"} = Ker\left(R \to (R/I)/\omega_\infty^\perp\right)$$

Ex: $\langle x^2y - y^2x, x^3, y^3, z^3 \rangle \subseteq \Bbbk[x,y,z]$

socle generator $\cancel{(x^2y-y^2x)}(x^2-xy+y^2)z^2$

$$Irr(I) = I + \cancel{(x^2y-y^2x)}(x^2-xy+y^2)$$

Irreducible decomposition arises from irreducible closures of coprincipal components at (essential) key witnesses. $\square$