Real applied multigraded commutative algebra

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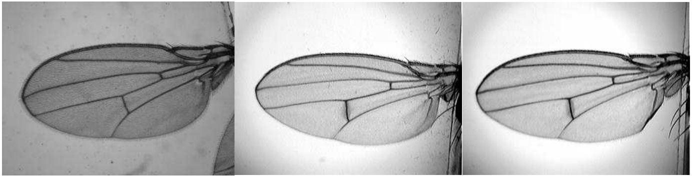
Outline

1. Fly wings
2. Biological background
3. Persistent homology
4. Multiparameter persistence
5. Poset modules
6. Next up

Fruit fly wings

Normal fly wings [images from David Houle’s lab]:



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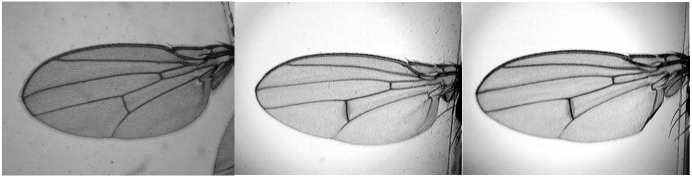
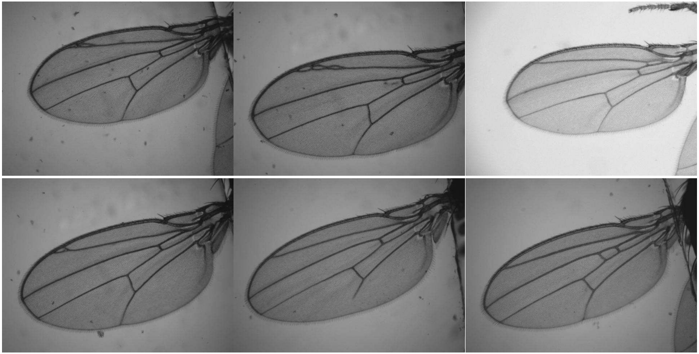


Topologically abnormal veins:



Biological background

What generates topological novelty?

[Houle, et al.]: selecting for certain continuous wing vein deformations yields

- skew toward more oddly shaped wings, but also
- much higher rate of topological novelty

Hypothesis. Topological novelty arises when directional selection pushes continuous variation in a developmental program beyond a certain threshold.

Test the hypothesis

- "plot" wings in "form space"
- determine whether topological variants lie "in the direction of" continuous shape selected for, and at the extreme in that direction

Goal. Statistical analysis encompassing topological vein variation, giving appropriate weight to new singular points in addition to varying shape

- compare phenotypic distance to genotypic distance; needs
- metric specifying distance between topologically distinct wings

To proceed. Statistics with fly wings as data objects ⇝ statistics with multiparameter persistence diagrams as data objects

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To proceed. Statistics with fly wings as data objects $\rightsquigarrow$ statistics with $\underbrace{\text{multiparameter persistence diagrams}}_{= \text{ modules over posets}}$ as data objects

Persistent homology

Topological space X

- Fixed $X \rightsquigarrow$ homology $H_i X$ for each dimension i
- Build X step by step: measure evolving topology

Def. Let $X_\bullet$ be a filtered space, meaning $\varnothing = X_0 \subset X_1 \subset \cdots \subset X_m = X$. The persistent homology $H_i X_\bullet$ is $H_i X_1 \to H_i X_2 \to \cdots \to H_i X_m$, a sequence of vector space homomorphisms.

Examples

1. Given a function $f : X \to \mathbb{R}$, let $X_k = f^{-1}\big((-\infty, t_k]\big)$. Good choice of $t_0, \ldots, t_m \in \mathbb{R}$: the values of t across which $H_i X_t$ changes
2. Any simplicial complex: build it simplex by simplex in some order

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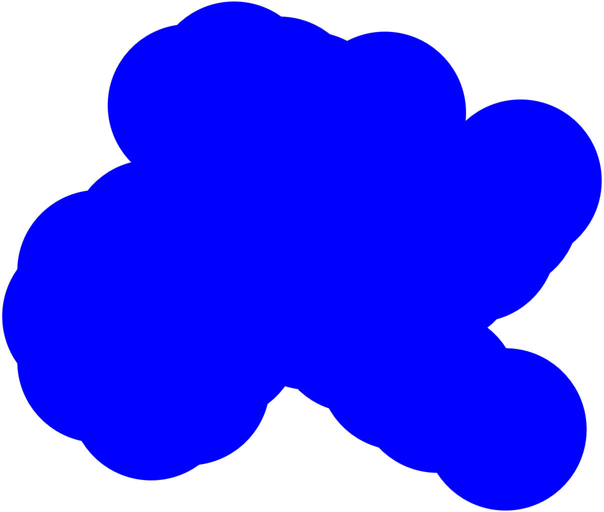
Example: expanding balls



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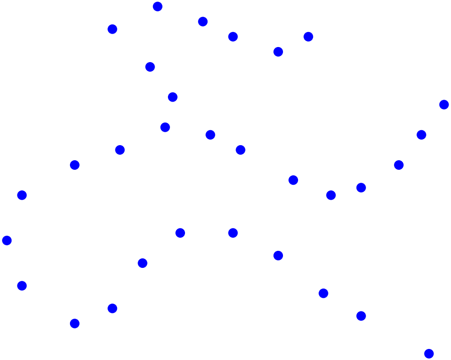

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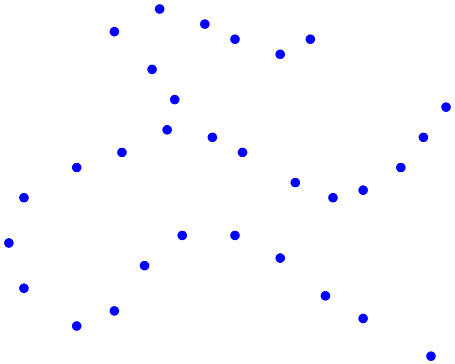
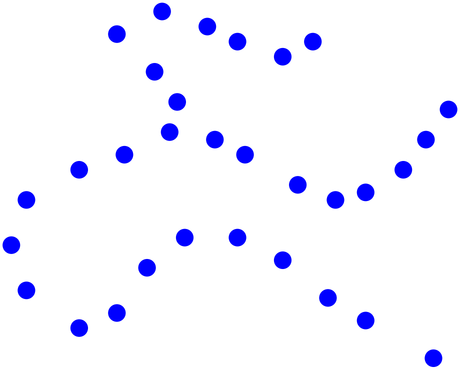


$\dim(H_0) = 31$

Example: expanding balls



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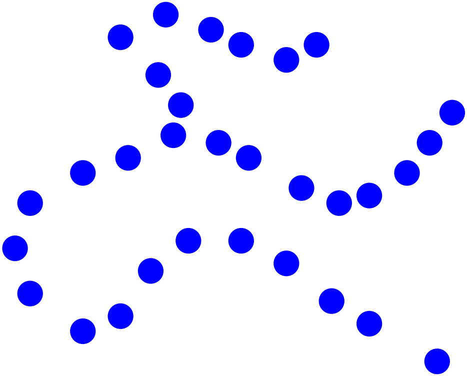

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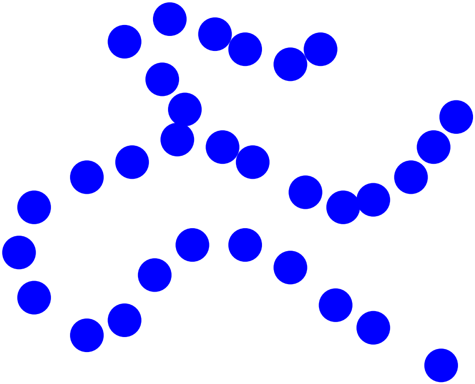

$\dim(H_0) = 26$

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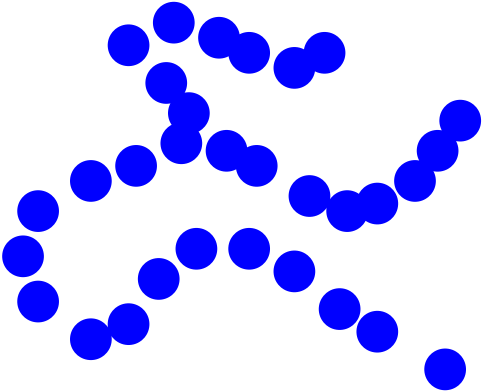
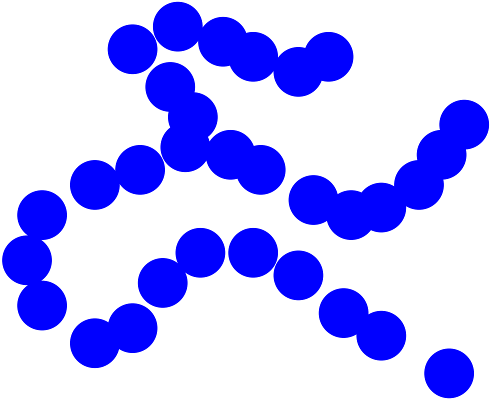


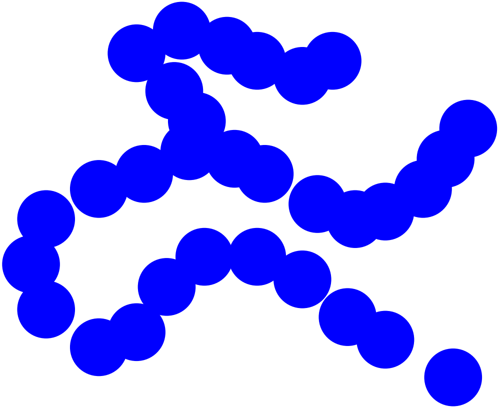
$\dim(H_0) = 21$

Example: expanding balls



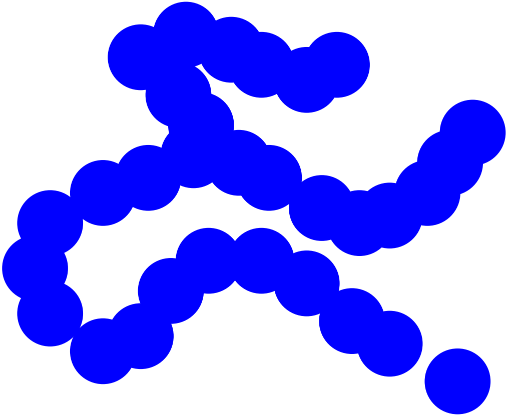
$\dim(H_0) = 12$

Example: expanding balls



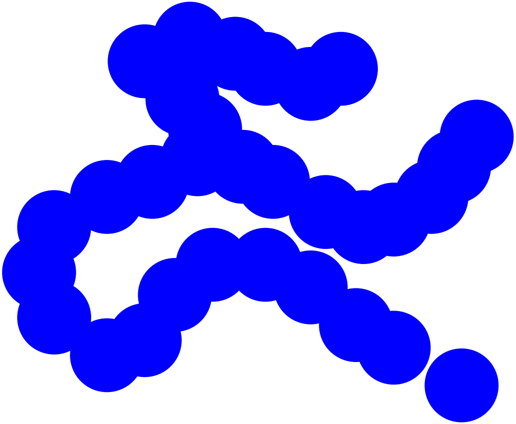
$\dim(H_0) = 6$

Example: expanding balls



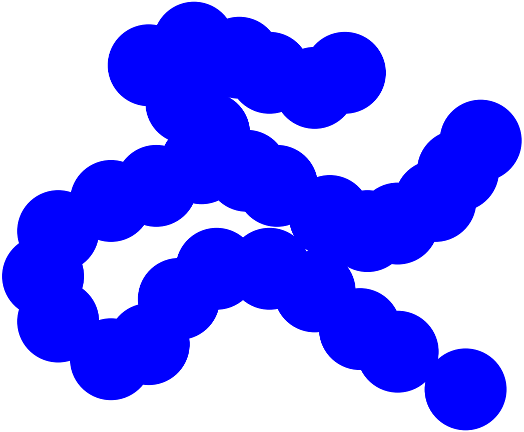
$\dim(H_0) = 2$

Example: expanding balls



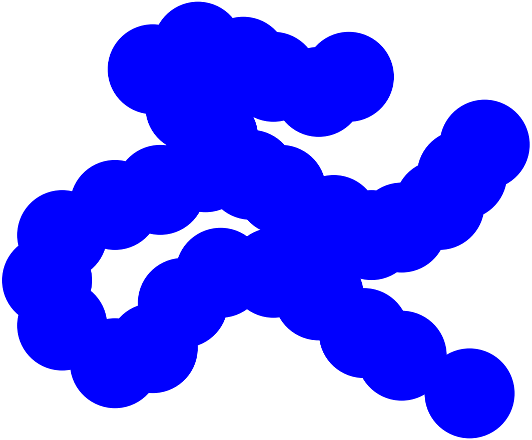
$\dim(H_0) = 2$

Example: expanding balls



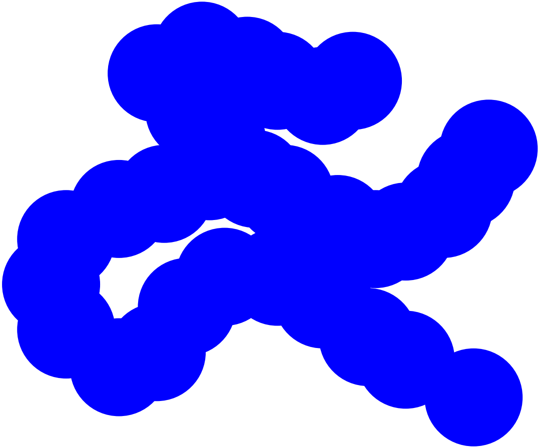
$\dim(H_0) = 1 \qquad \dim(H_1) = 2$

Example: expanding balls



$\dim(H_0) = 1 \qquad \dim(H_1) = 1$

Example: expanding balls



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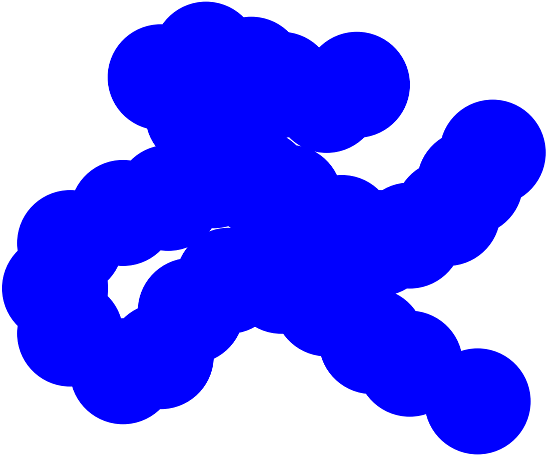
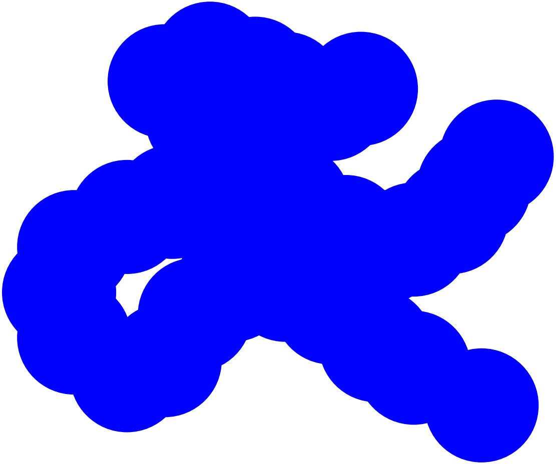


$\dim(H_0) = 1 \qquad \dim(H_1) = 3$

Example: expanding balls



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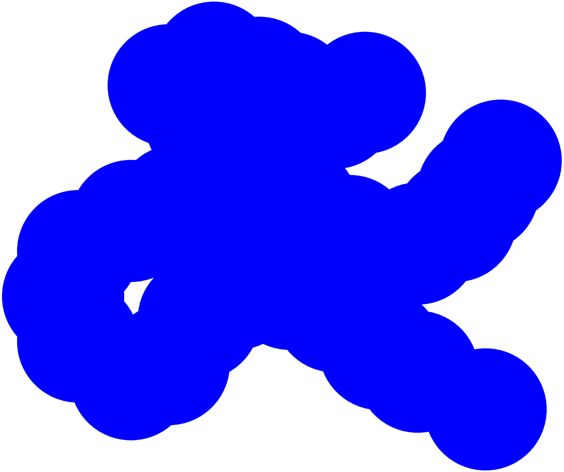

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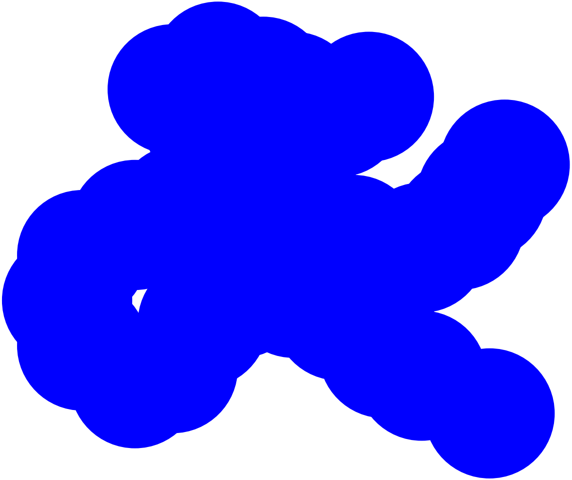

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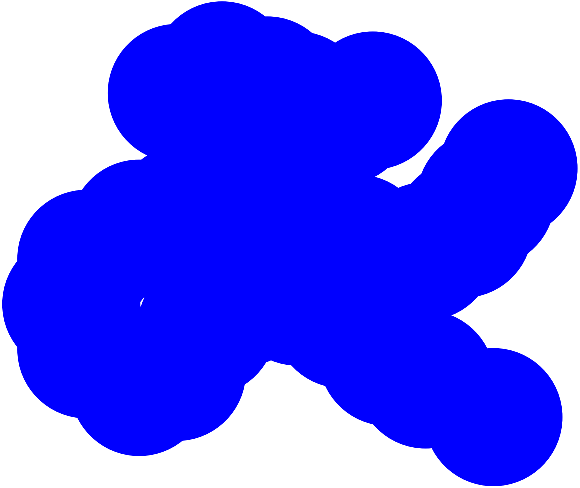

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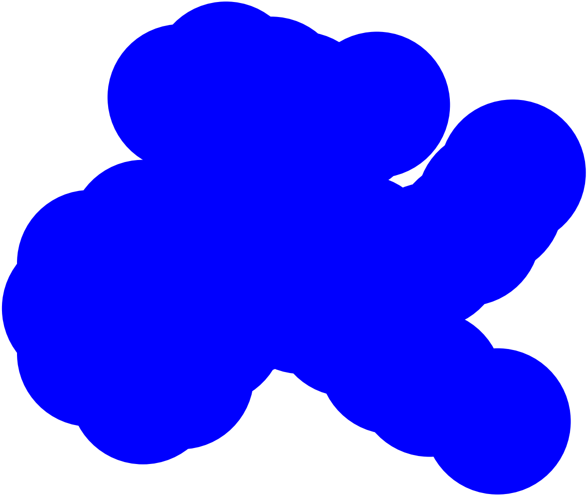

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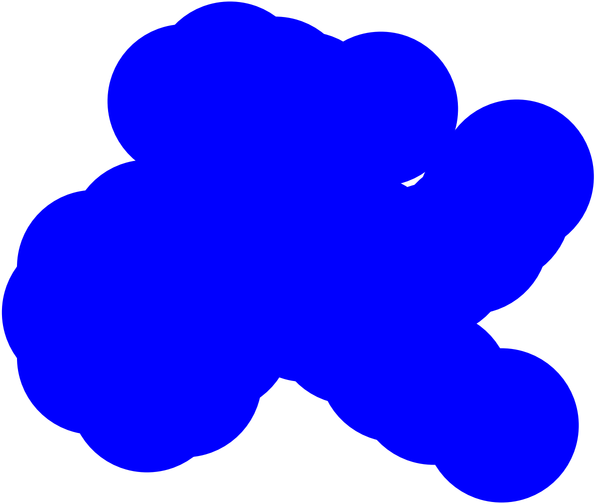

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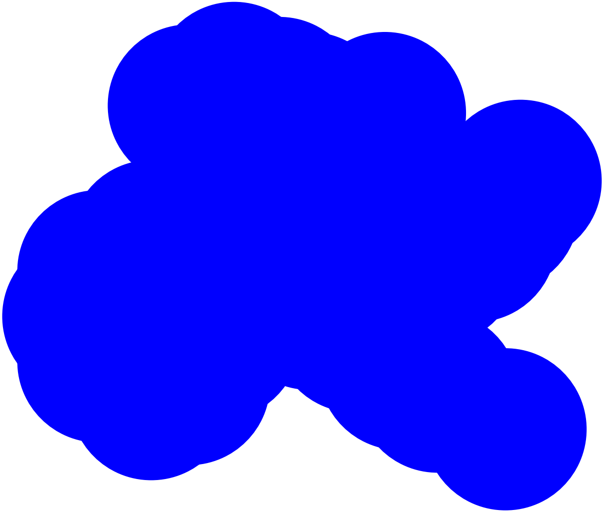

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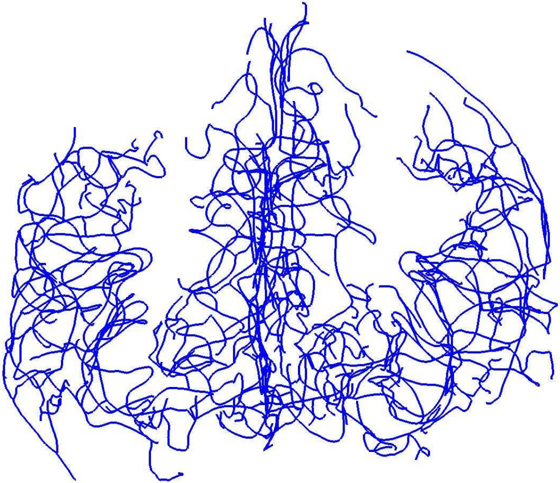
Example: filling brains [w/Bendich, Marron, Pieloch, Skwerer]



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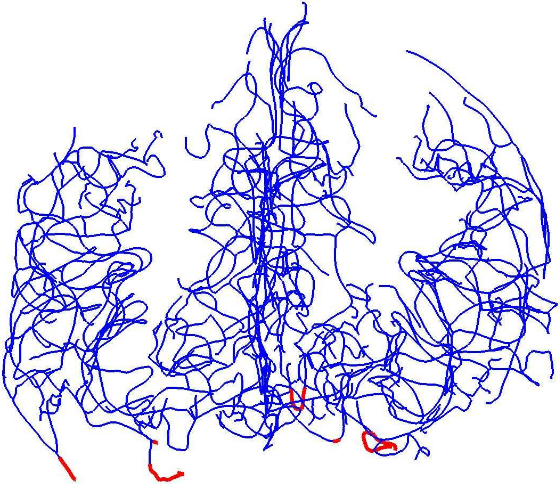

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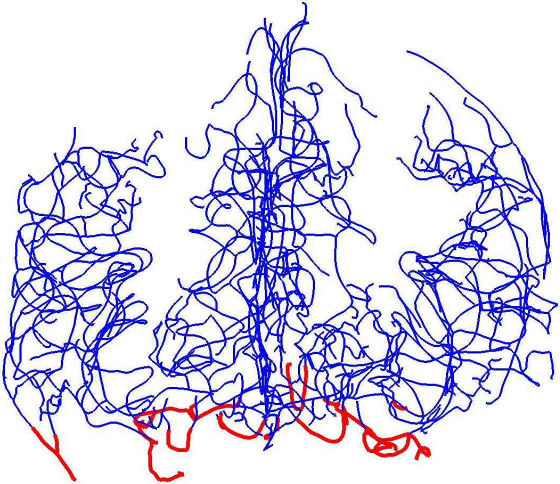

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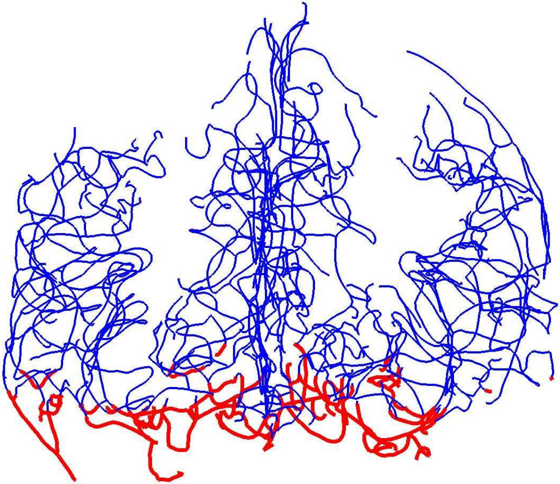

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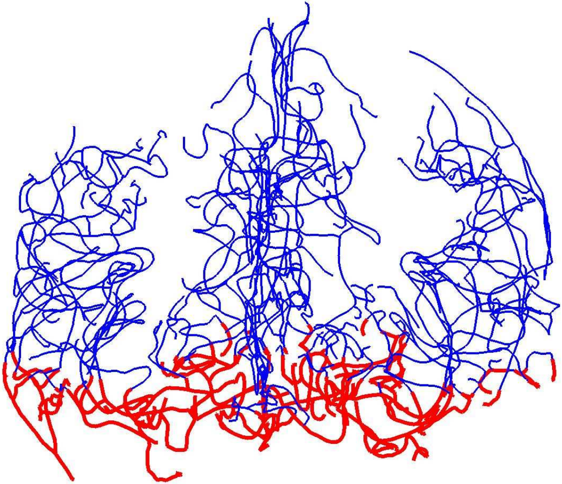

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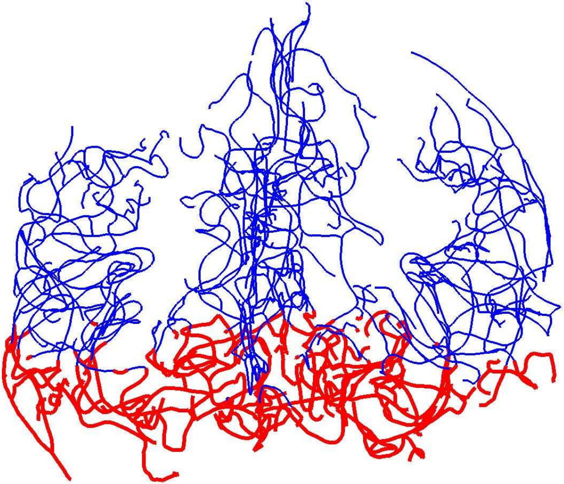

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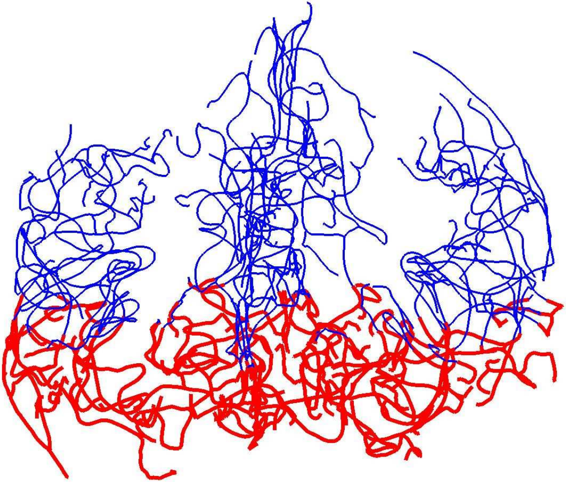

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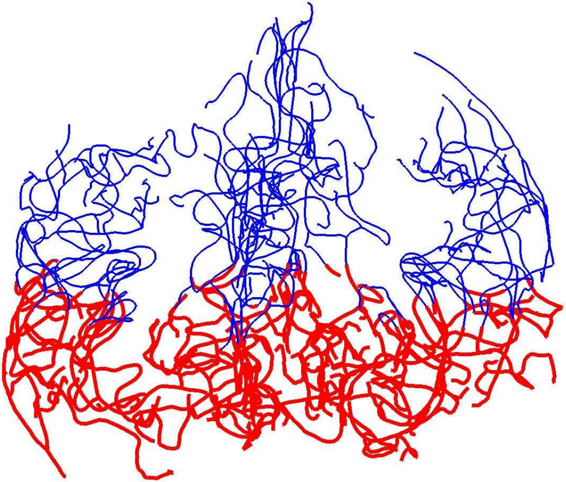

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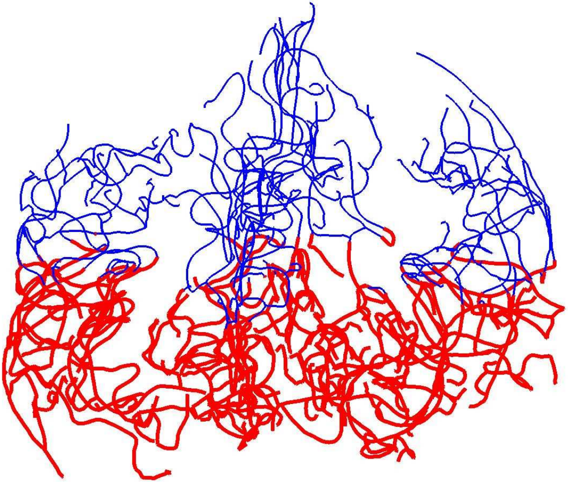

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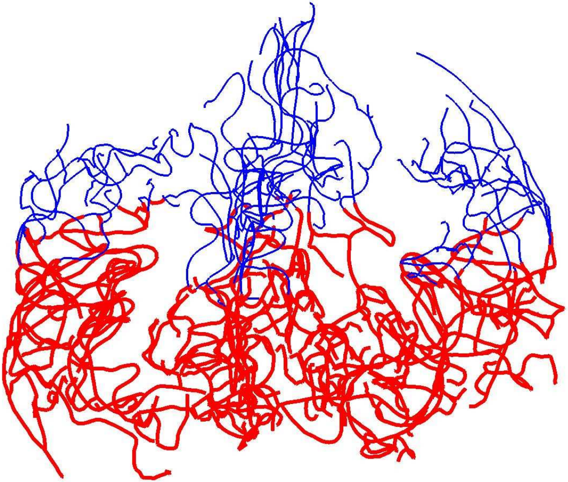

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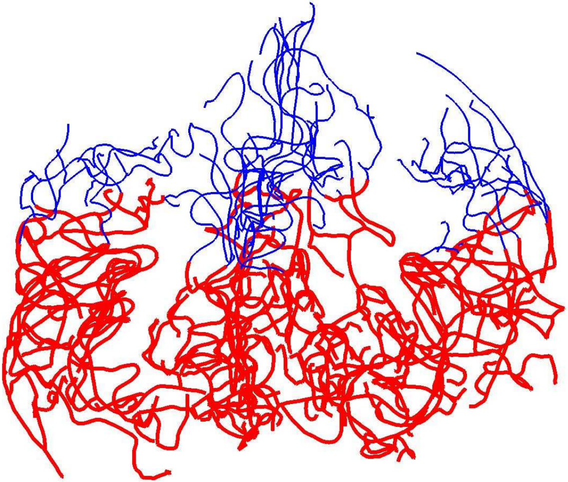

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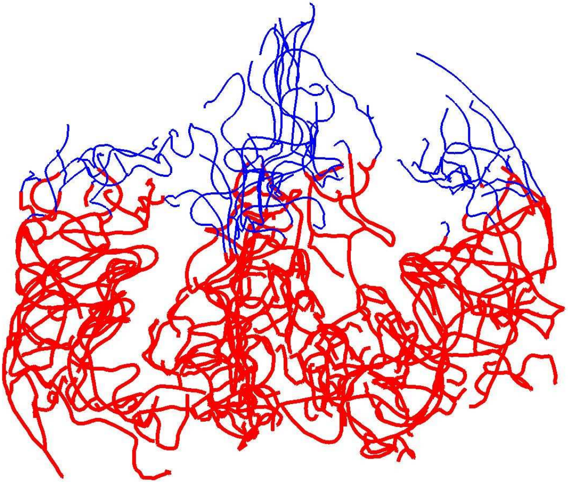

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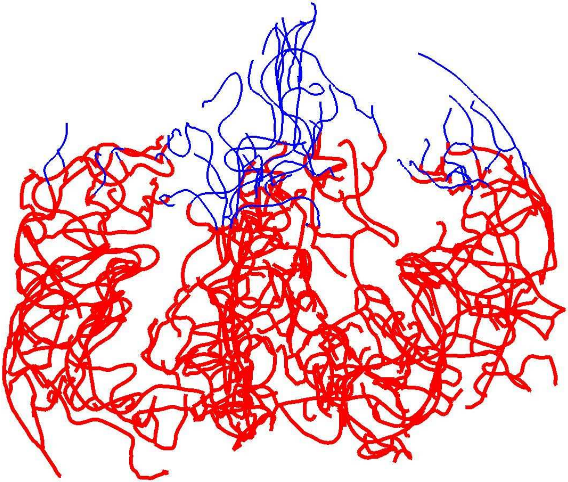

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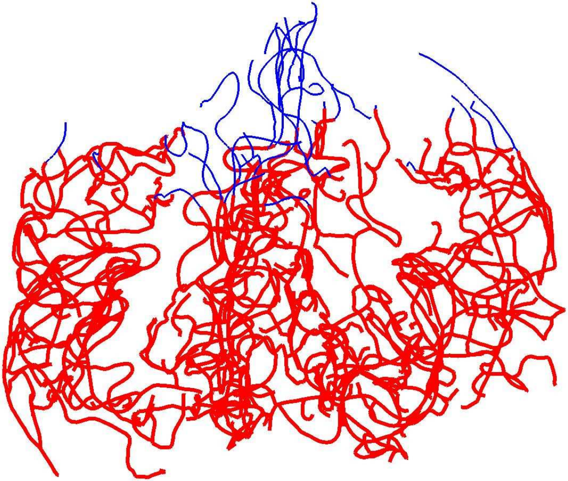

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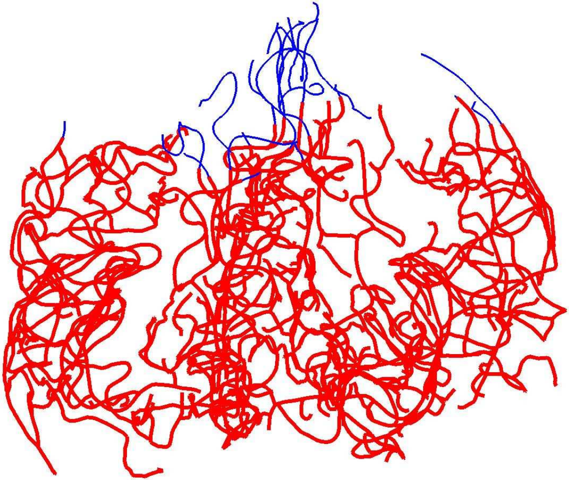

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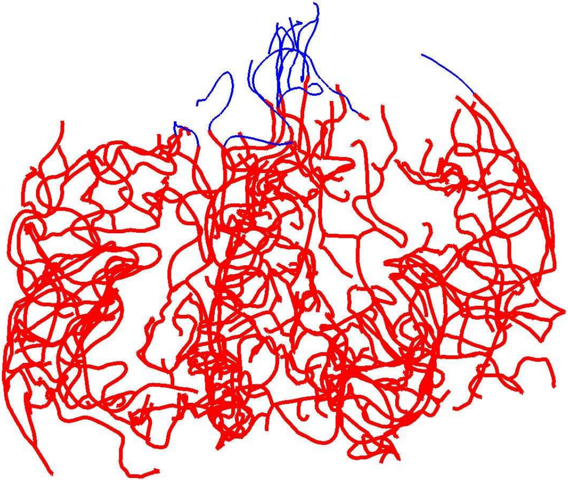

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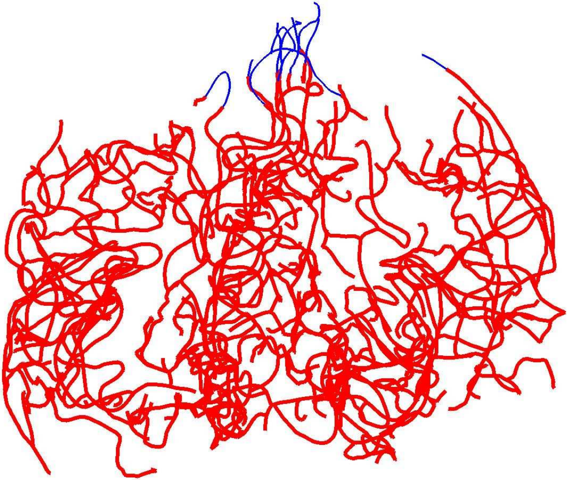

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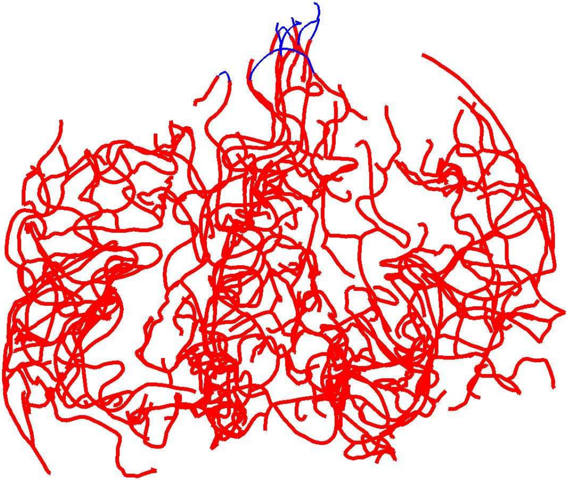

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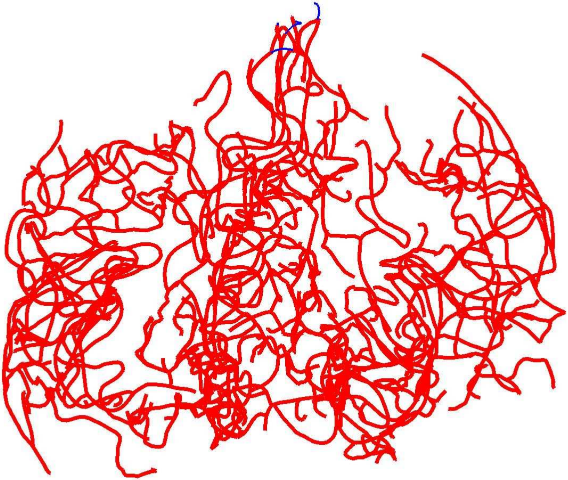

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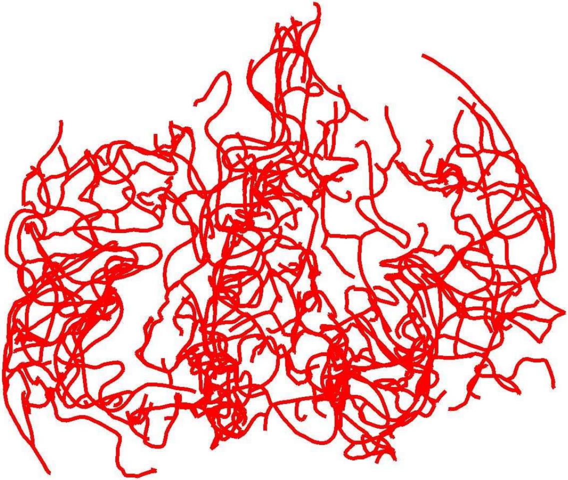

Persistent homology

Topological space X

- Fixed $X \rightsquigarrow$ homology $H_i X$ for each dimension i
- Build X step by step: measure evolving topology

Def. Let $X_\bullet$ be a filtered space, meaning $\varnothing = X_0 \subset X_1 \subset \cdots \subset X_m = X$. The persistent homology $H_i X_\bullet$ is $H_i X_1 \to H_i X_2 \to \cdots \to H_i X_m$, a sequence of vector space homomorphisms.

Examples

1. Given a function $f : X \to \mathbb{R}$, let $X_k = f^{-1}\big((-\infty, t_k]\big)$. Good choice of $t_0, \ldots, t_m \in \mathbb{R}$: the values of t across which $H_i X_t$ changes
2. Any simplicial complex: build it simplex by simplex in some order

History. invented by [Frosini, Landi 1999], [Robins 1999];
[Edelsbrunner, Letscher, Zomorodian 2002]: includes efficient computation;
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Multiparameter persistence

Plan. (with Houle, Curry, Thomas, +. . .) Encode with 2-parameter persistence

- 1st parameter: distance from vertex set
- 2nd parameter: distance from edge set

Sublevel set $W_{r,s}$ is near edges but far from vertices

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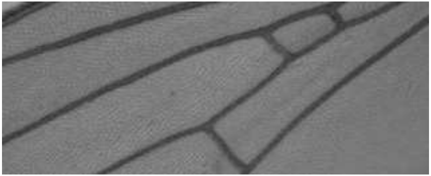

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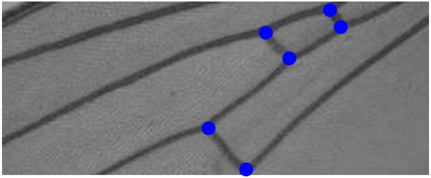

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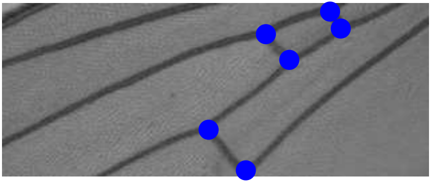

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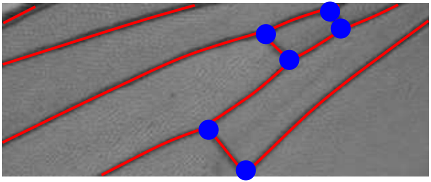

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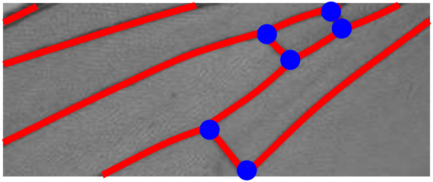

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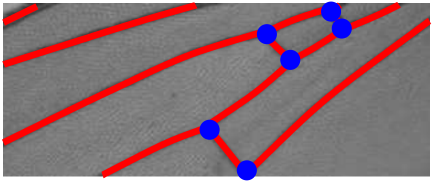

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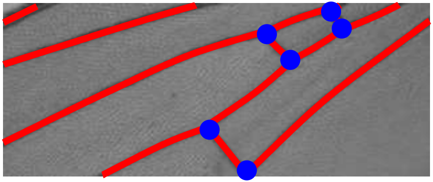

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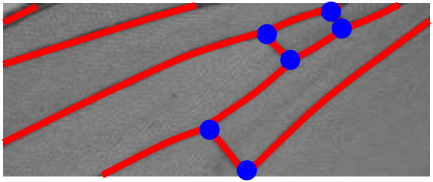

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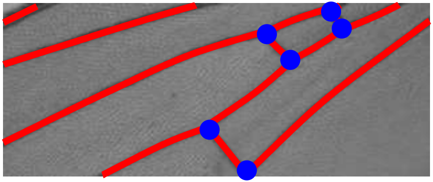

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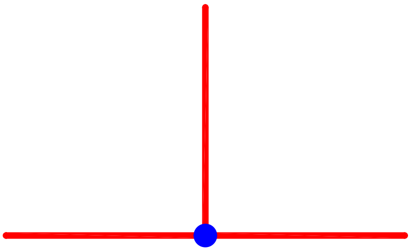
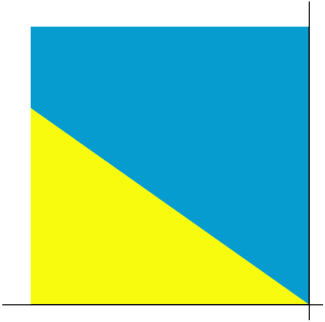


A piece of fly wing vein

$\rightsquigarrow$



The (r, s)-plane $\mathbb{R}^2$

Observations

- stratification alters persistence module
- discretization approximates something algebraic

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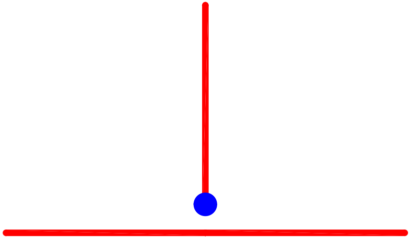

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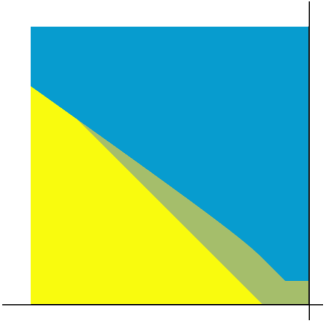

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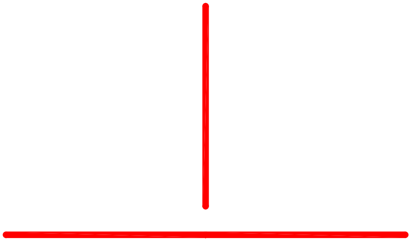

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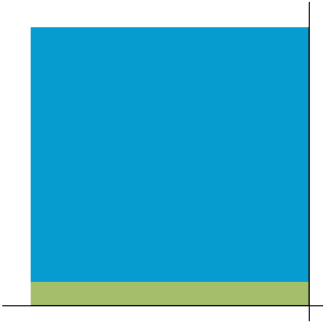

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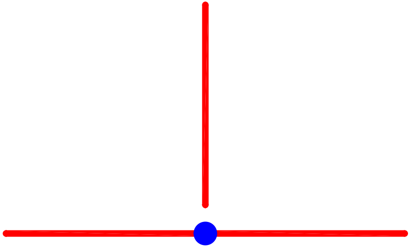

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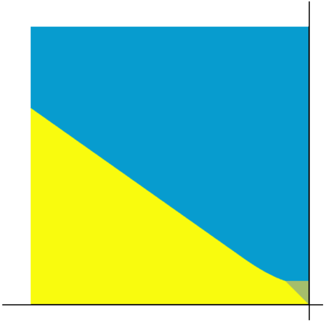

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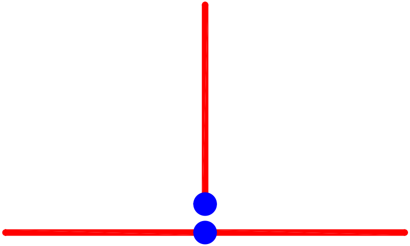
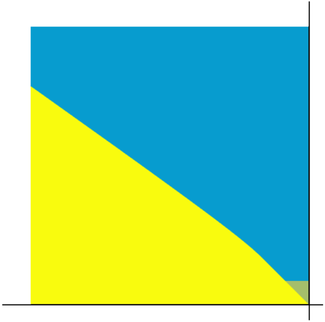


A piece of fly wing vein

$\rightsquigarrow$



The (r, s)-plane $\mathbb{R}^2$

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- stratification alters persistence module
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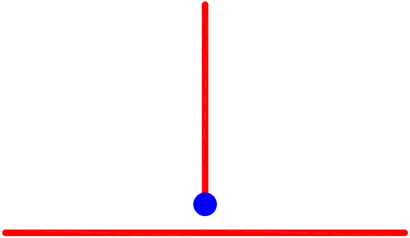

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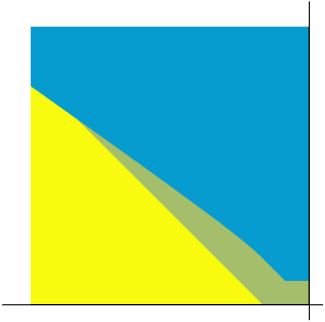

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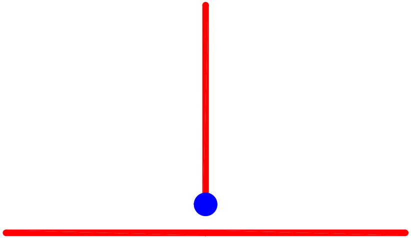

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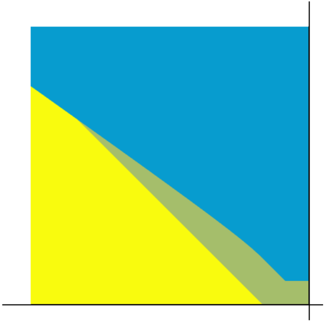

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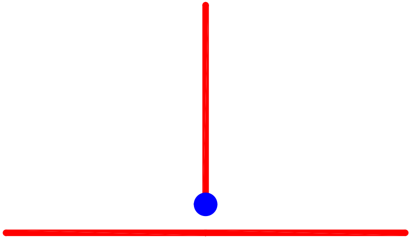

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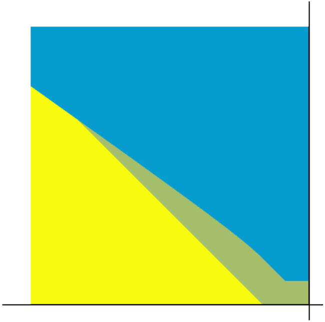

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Modules over posets

Def. partially ordered set $(Q, \preceq)$: relation $\preceq$ is

- reflexive: $q \preceq q$
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- Q-graded vector space $H = \bigoplus_{q \in Q} H_q$ with
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& {} & {} & {} & \\
{} \arrow[r, "\cdot x"] & H_{r-\varepsilon,s+\delta} \arrow[u, "\cdot y"] \arrow[r, "\cdot x"] & H_{r,s+\delta} \arrow[u, "\cdot y"] \arrow[r, "\cdot x"] & H_{r+\varepsilon,s+\delta} \arrow[u, "\cdot y"] \arrow[r, "\cdot x"] & {} \\
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& {} \arrow[u, "\cdot y"] & {} \arrow[u, "\cdot y"] & {} \arrow[u, "\cdot y"] &
\end{tikzcd}$$

Next up

1. modules, gradings, and topology from statistical problems in biology
2. upsets and downsets: free, flat, and injective modules
3. resolutions of $\mathbb{Z}^n$-graded modules over $\Bbbk[x_1, \ldots, x_n]$
4. poset encoding: lift homological algebra to modules over poests
    - syzygy theorem
    - fringe presentation: data structure for persistent homology
5. how to do statistics on sets of peristence modules
    - moduli problems
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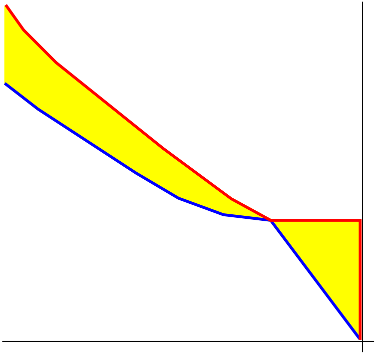
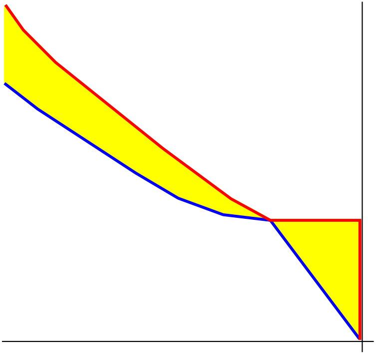


An $\mathbb{R}^2$-module

$\rightsquigarrow$



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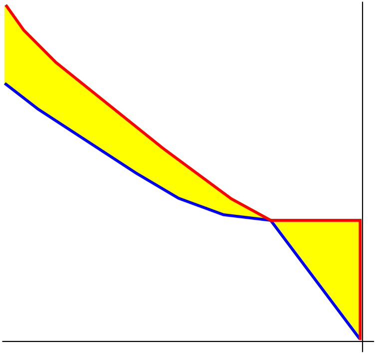

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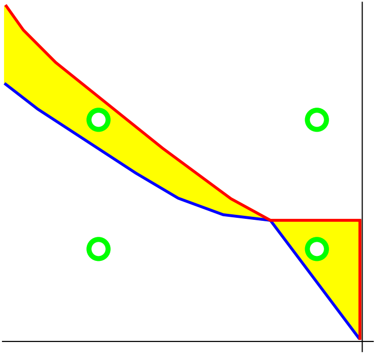

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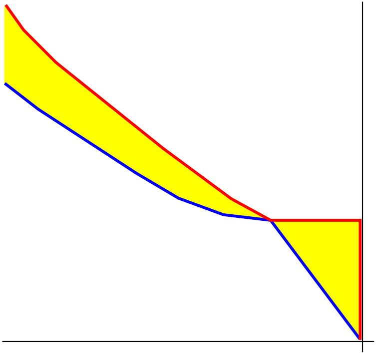

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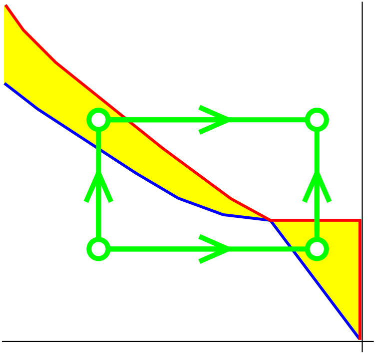

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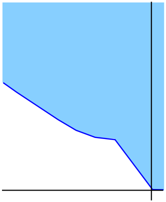

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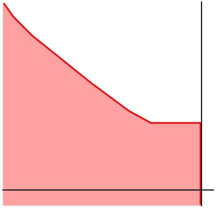

$= D$

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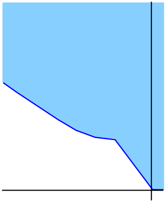
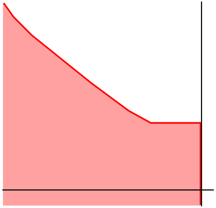
$U =$  and  $= D$

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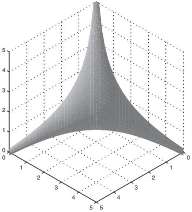
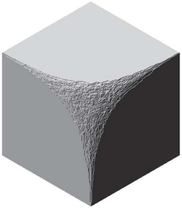
$U =$ 

or  $= D$

semialgebraic piecewise linear

[Andrei Okounkov, *Limit shapes, real and imagined*, Bulletin of the AMS **53** (2016), no. 2, 187–216]

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birth upsets →

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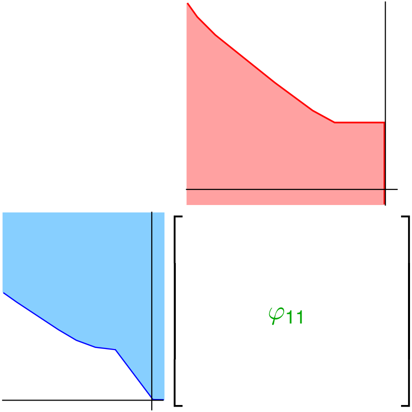
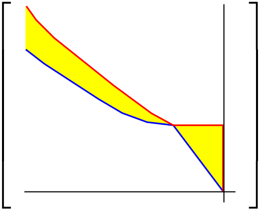


$\left[\ \varphi_{11}\ \right]$ has image $\Bbbk$



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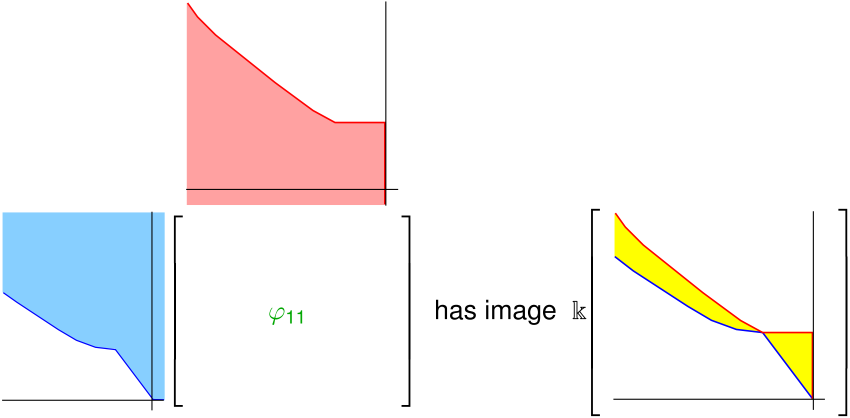

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For any subset $S \subseteq Q$, set $\Bbbk[S] = \bigoplus_{s \in S} \Bbbk_s$.

Def [w/A. Thomas]. A fringe presentation of H is a monomial matrix

birth upsets →

$$\begin{array}{cc} & \begin{array}{ccc} D_1 & \cdots & D_\ell \end{array} \\ \begin{array}{c} U_1 \\ \vdots \\ U_k \end{array} & \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1\ell} \\ \vdots & \ddots & \vdots \\ \varphi_{k1} & \cdots & \varphi_{k\ell} \end{bmatrix} \end{array}$$

← death downsets

← scalar entries

$$\Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k] = F \xrightarrow{\qquad\qquad\qquad} E = \Bbbk[D_1] \oplus \cdots \oplus \Bbbk[D_\ell]$$

with image$(F \to E) \cong H$.

- subordinate to encoding $\pi : Q \to P$ if all U_i and D_j are unions of fibers of π

Compare. [Chachólski, Patriarca, Scolamiero, Vaccarino] “monomial presentation”

Fringe presentation

Def [w/A. Thomas]. H has finite encoding $\pi : Q \to P$ if

- P finite poset and
- $H \cong \pi^* M = \bigoplus_{q \in Q} H_{\pi(q)}$, the pullback of M along π.

For $Q = \mathbb{R}^n$, encoding is semialgebraic if its fibers are semialgebraic varieties.

Def. Fix a poset Q.

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