Lecture 1: see slides
Lectures 2–5: handwritten notes

Real applied multigraded commutative algebra

24th National School on Algebra
EMS Summer School on Multigraded Algebra & Applications
Mistral Resort, Moieciu de Sus, Romania
17–24 August 2016, Thursday 11:10 – 12:00
Friday 9:00 – 9:50
Saturday 11:10 – 12:00, 12:10 – 13:…
Monday 10:00 – 10:50

$x = x_1, \dots, x_n$

$\Bbbk$ field

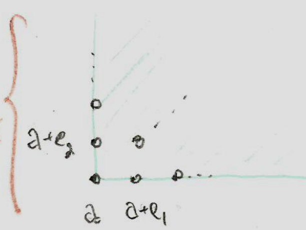
$\Bbbk[x] = \bigoplus_{b \in \mathbb{N}^n} \Bbbk \cdot x^b$ $\quad \mathbb{Z}^n$-graded

module M $\mathbb{Z}^n$-graded if $M = \bigoplus_{a \in \mathbb{Z}^n} M_a$ $\quad \Leftrightarrow$ M is $\mathbb{Z}^n$-module$/\Bbbk$.

$x^b : M_a \to M_{a+b}$.

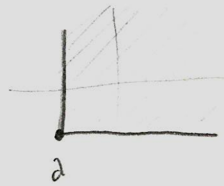
<u>M free$/\Bbbk[x]$</u>

rank 1: basis $e \in M_a$

$M = \Bbbk \left\{ \right.$  $\left. \subseteq \mathbb{Z}^n \right\} = \Bbbk[\mathbb{Z}^n_{\succeq a}]$

$\Rightarrow$ $M \hookrightarrow \Bbbk[\mathbb{Z}^n]$

$\Bbbk[x] \cdot x^a = \Bbbk[x](-a)$



how you're probably used to seeing it

rank ≥ 1: $\bigoplus$ rank 1.

<u>localization</u> $M[f^{-1}]$ $\mathbb{Z}^n$-graded needs f homogeneous

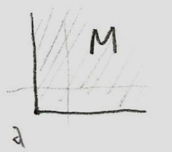
$\Leftrightarrow$ $f = x^b$ some $b \in \mathbb{N}^n$

may as well $b \in \{0,1\}^n \leftrightarrow \sigma \subseteq \{1, \dots, n\} = [n]$.

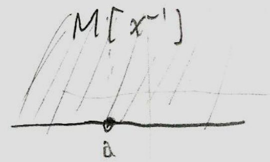
M has only 2^n $\mathbb{Z}^n$-graded localizations $M[x^{-\sigma}]$

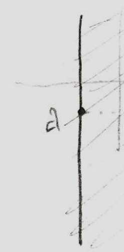
e.g. $M = \Bbbk[\mathbb{Z}^n_{\succeq a}]$ $\Rightarrow$ $M[x^{-\sigma}] = \Bbbk[a + \mathbb{Z}^\sigma \times \mathbb{N}^{\bar\sigma}]$

$= \Bbbk[a + \mathbb{N}^n]$



$\rightsquigarrow$





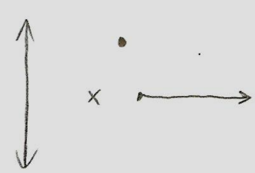
[LocalDuality, 3.5]

<u>Lemma</u>: $M \cong \bigoplus$ localizations of free mods

$\Downarrow$ $\quad \Uparrow$ if M is $\mathbb{Z}^n$-finite: $\dim_\Bbbk M_a < \infty \; \forall a \in \mathbb{Z}^n$

M flat

(in $\mathbb{Z}^n$-mod $\Rightarrow$ in $\Bbbk[x]$-mod)



Matlis duality

intuitively: turn M upside-down

precisely: Def: The Matlis dual of M is $M^\vee$ with

$$(M^\vee)_a = (M_{-a})^* \qquad (-)^* = \mathrm{Hom}_{\Bbbk}(-,\Bbbk) \text{ vector space dual.}$$

Prop$^\vee$: $M \rightsquigarrow M^\vee$ is exact

$M \xrightarrow{\sim} (M^\vee)^\vee$ if M $\mathbb{Z}^n$-finite.

M flat $\Leftrightarrow$ $M^\vee$ injective

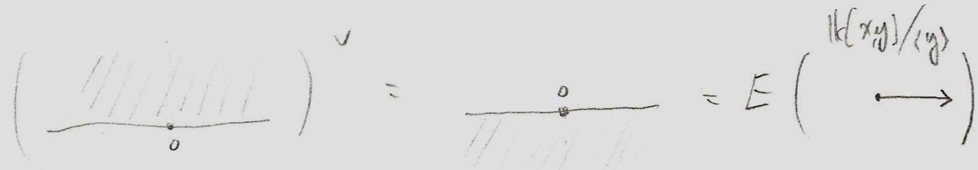
($M\otimes -$ exact) $\quad$ ($\mathrm{Hom}(-,M)$ exact)

Pf: Exercise!

Def: indecomposable injective

e.g. $\Bbbk[x][x^{-\sigma}]^\vee = \begin{pmatrix} \Bbbk[\mathbb{Z}^\sigma \times \mathbb{N}^{\bar\sigma}] \\ \downarrow \\ \Bbbk[\mathbb{Z}^n] \end{pmatrix}^\vee = ?\ \begin{matrix} \Bbbk[\mathbb{Z}^\sigma \times -\mathbb{N}^{\bar\sigma}] \\ \uparrow \\ \Bbbk[\mathbb{Z}^n] \end{matrix} = E(\Bbbk[x]/P_\sigma) = E(\Bbbk[\mathbb{N}^\sigma])$.

$P_\sigma = \langle x_i \mid i \notin \sigma \rangle$.



Prop [GW '78] $M \cong \bigoplus$ indec. injectives

$\Updownarrow$ $\quad$ (no $\mathbb{Z}^n$-finiteness required.)

M injective (in $\mathbb{Z}^n$-mod)

Pf: Same as for R-mod, R noeth.

Cor: Lemma

Pf: Prop$^\vee$: M flat $\Rightarrow M^\vee$ inj $\Rightarrow M^\vee \cong \bigoplus \Rightarrow M^{\vee\vee} = \prod$

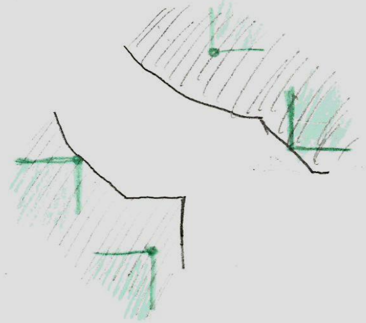
$\cong M$ if $\mathbb{Z}^n$-finite. $\square$

Analogues for arbitrary posets Q? e.g. Q finite ($\Rightarrow$ flat is hard to do)

$Q = \mathbb{R}^n$ ($\Rightarrow$

Def: $U \subseteq Q$ <u>upset</u> if $Q_{\succeq u} \subseteq U \ \forall u \in U$

$D \subseteq Q$ <u>downset</u> if $Q_{\preceq d} \subseteq D \ \forall d \in D$.



We'll be decomposing Q-mods using these instead of free, flat, or inj.

$S \subseteq Q \Rightarrow \Bbbk[S] = \bigoplus_{s \in S} \Bbbk_s$

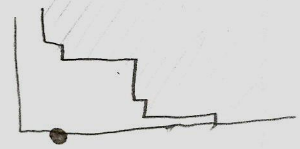
Lemma: $\Bbbk[U] \subseteq \Bbbk[Q]$ <u>indicator submodule</u>

$\Bbbk[Q] \twoheadrightarrow \Bbbk[D]$. <u>indicator quotient module</u>

e.g. indec. free or flat $\Rightarrow$ indicator submod.

" injective $\Rightarrow$ " quotient

monomial ideal in $\Bbbk[x]$ = indicator submod of upset in $\mathbb{N}^n$



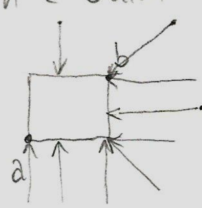
To lift from $\mathbb{Z}^n$ or $\mathbb{Q}$ to $\mathbb{R}^n$ or Q, need:

Def: $\pi : Q \to P$ is a poset <u>morphism</u> if $q \preceq q' \Rightarrow \pi(q) \preceq \pi(q')$. cf. Jürgen's lecture yesterday

e.g. • Q finite $\hookrightarrow \mathbb{Z}^n = P$ n minimal $\Rightarrow n = \operatorname{odim} Q$

• $Q = \mathbb{Z}^n \twoheadrightarrow P = [a, b]$ a box

convex projection



• upset $U = \pi^{-1}(1 \in \{0<1\})$

• downset $D = \pi^{-1}(0 \in \{0<1\})$

Lemma: $\pi^{-1}(\text{upset}) = \text{upset}$

$\pi^{-1}(\text{downset}) = \text{downset}$ cf. Jürgen's lecture yesterday

Ex: $\operatorname{Hom}_{Q\text{-mod}}(\Bbbk[U], \Bbbk[U']) = ?$

$\Bbbk[D], \Bbbk[D]$

$\Bbbk[U], \Bbbk[D]$

$\Bbbk[D], \Bbbk[U]$

DO ON 5

Monomial matrices

objects → column labels, row labels

$$\begin{array}{c|cccc|} & B_1 & \cdots & B_\ell \\ \hline A_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\ \vdots & & \text{scalar entries} & \vdots \\ A_k & \varphi_{k1} & \cdots & \varphi_{k\ell} \\ \hline \end{array}$$

monomial matrix representing if $\varphi_{ij} = 0$ whenever Hom dictates it has to

$$A_1 \oplus \cdots \oplus A_k \longrightarrow B_1 \oplus \cdots \oplus B_\ell \qquad \text{if } \operatorname{Hom}(A_i, B_j) = \Bbbk \text{ or } 0$$

$$A \longleftarrow B \qquad B_j,\ A_i$$

DO ON 5

e.g. • $A \to B$ both $\mathbb{Z}^n$-finite injective $A_i = \Bbbk[D_i]$, $B_j = \Bbbk[D_j]$

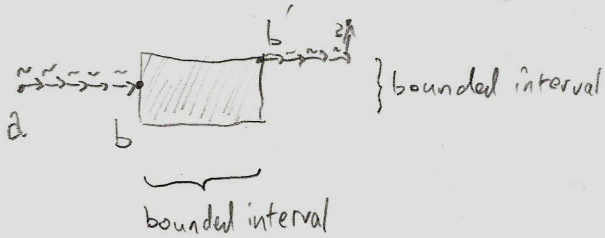
• $A \leftarrow B$ " " flat U_i U_j

• $A \to B$ or $A_i = \Bbbk[U_i]$ $B_j = \Bbbk[D_j]$

flat injective

Def: $\mathbb{Z}^n$-finite M is <u>finitely determined</u> if $\forall\, i = 1, \ldots, n$

$$\cdot x_i : M_a \xrightarrow{\sim} M_{a+e_i}$$



whenever $a_i \notin$ some bounded interval.

"Everything about M is determined by what happens inside of a box"

e.g. M f.g.: all gens $\succeq b$

all relations $\preceq b'$.

e.g. ⊕ localizations of $\Bbbk[x](-a)$ = $\mathbb{Z}^n$-finite flat

e.g. M f.d. $\Leftrightarrow$ $M^\vee$ f.d.

e.g. $\mathbb{Z}^n$-finite injective

Def: Fix $\mathbb{Z}^n$-mod M.

1. injective hull (envelope) of M is $M \hookrightarrow E$ with E injective
   - finite if E $\mathbb{Z}^n$-finite
   - minimal if # summands of E is minimal.
2. injective resolution of M is a complex
$$E^\bullet : 0 \to E^0 \to E^1 \to \cdots \to E^j \to \cdots$$
of injectives E^j with $H^j E^\bullet = \begin{cases} M & j=0 \\ 0 & j \geq 1 \end{cases}$.
   - length ℓ if $E^j = 0$ for $j > \ell$ and $E^\ell \neq 0$.
   - finite if $E^\bullet$ is $\mathbb{Z}^n$-finite
   - minimal if # summands of E^j is minimized $\forall j$. $\Leftrightarrow$ $M \to E^0$ and $\operatorname{coker}(E^{j-1} \to E^j) \to E^{j+1}$ are injective hulls $\forall j \geq 1$.

Thm f.d.: M f.d. $\Leftrightarrow$ M has finite minimal inj. res. of length $\leq n$, unique up to $\cong$.

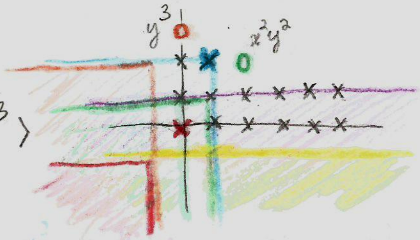
Pf: Basically [GW'78]. Essential point: $\operatorname{ann}(y) = \mathfrak{p}_\sigma$ ($y \in M_a$) $\Rightarrow M \xrightarrow{\psi} \Bbbk[a + \mathbb{Z}^\sigma - \mathbb{N}^{\bar\sigma}]$ with $\psi(y) \neq 0$. Primary decomp. (of principal submods) $\Rightarrow$ every elt. of M has multiple that is one of finitely many such cogenerators. $\square$

[$z \in M_a \Rightarrow z' \in M_{a'}$ with "$z \cong z'$" and $a' \in [b, b']$; $M_{[b,b']}$ f.g.]

Cor: Same for flat resolutions!

Pf: $(\text{Thm})^\vee$.

E.g. $M = \Bbbk[x,y]/\langle x^2y^2, y^3 \rangle$



$$0 \to \Bbbk[(1,2) - \mathbb{N}^2] \oplus \Bbbk[(0,1) - \mathbb{Z}\times\mathbb{N}] \longrightarrow \Bbbk[(1,1) - \mathbb{N}^2] \oplus \Bbbk[(0,-1) - \mathbb{Z}\times\mathbb{N}] \oplus \Bbbk[(-1,2) - \mathbb{N}^2] \longrightarrow \Bbbk[(-1,-1) - \mathbb{N}^2] \to$$

Monomial matrices

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

answer

from p. 4

+ $\operatorname{Hom}_{Q\text{-mod}}(-,-) = \Bbbk$

from p. 3

Exercise: Matlis dual of this monomial matrix.

Poset encoding

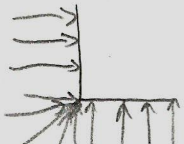
Def: Fix $\pi : Q \to P$ and P-mod M. The pullback of M along π is

Q-mod $\pi^* M$, which has $(\pi^* M)_q = M_{\pi(q)}$.

e.g. $\Bbbk[U] = \pi^* \left(\begin{smallmatrix} \Bbbk \\ 0 \end{smallmatrix} \right)$

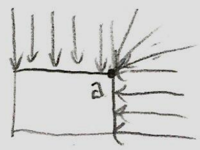
$\Bbbk[D] = \pi^* \left(\begin{smallmatrix} 0 \\ \Bbbk \end{smallmatrix} \right)$

$\mathbb{Z}^n \xrightarrow{\text{convex projection}} \mathbb{N}^n$



$\Rightarrow \pi^* M = \check{C} M$ Čech hull.

$\mathbb{N}^n \xrightarrow{\text{convex projection}} [0, a]$



$\Rightarrow \pi^* M = P_a M$ positive extension

e.g. M is squarefree if $M = P_{(1,\dots,1)} M$

M f.d. $\Leftrightarrow$ $M = \pi^* M_{[a,b]}$ for some $a \preceq b$

$\pi : \mathbb{Z}^n \to [a,b]$ convex projection

Important: insert Functoriality, p. 8

Def: Fix poset Q and Q-mod H.

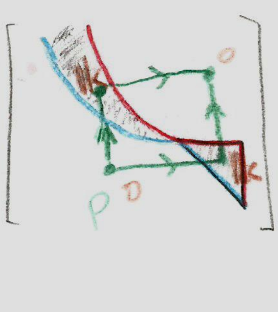
An encoding of H by a poset P is

- morphism $\pi : Q \to P$ and
- P-mod M with $H \cong \pi^* M$.

The encoding is finite if

- $|P| < \infty$ and
- $\dim_{\Bbbk} M_p < \infty$ $\forall\, p \in P$.

e.g. all the examples we just saw, but even more powerful as (finite) data structure for modules$/\mathbb{R}^n$.

$\Bbbk$ 

$\cong \pi^* H$

$\pi : \mathbb{R}^2 \to P$

Indicator resolutions

Def: Fix Q-mod. H.

1. $F_\bullet$ a <u>homological indicator resolution</u> of H if $\qquad F_\bullet : \cdots \leftarrow F_{i-1} \leftarrow F_i \leftarrow \cdots$
   - $F_i = \bigoplus(\text{indicator submods of } \Bbbk[Q]) \ \forall i$
   - $\cdots \leftarrow F_{i-1} \leftarrow F_i \leftarrow \cdots$
   - $H_i F_\bullet = \begin{cases} 0 & i \neq 0 \\ H & i = 0 \end{cases}$.

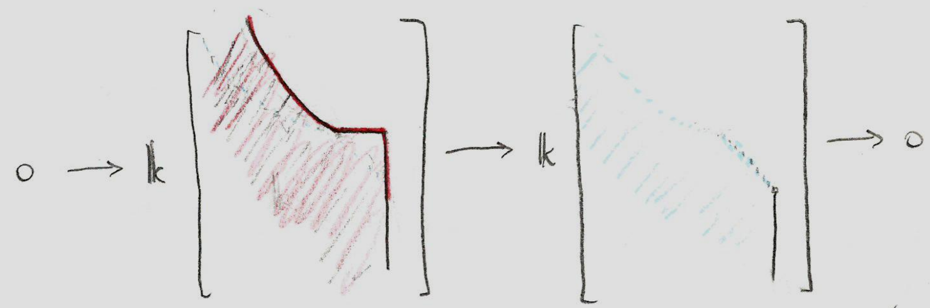
2. $E^\bullet$ a <u>cohomological indicator resolution</u> of H if $\qquad E_\bullet : \cdots \to E^j \to E^{j+1} \to \cdots$
   - $E^j = \bigoplus(\text{indicator quotients of } \Bbbk[Q]) \ \forall j$
   - $\cdots \to E^j \to E^{j+1} \to \cdots$
   - $H^j E^\bullet = \begin{cases} 0 & i \neq 0 \\ H & i = 0 \end{cases}$.

3. <u>length</u>: $\max\{i \mid F_i \neq 0\}$
   $\max\{j \mid E^j \neq 0\}$.

4. <u>finite</u> if # indicator summands $< \infty$ $\Rightarrow$ length $< \infty$.

5. <u>subordinate</u> to an encoding of H if each summand is constant on every fiber of the encoding $\Big\} \Leftrightarrow$ each summand is pulled back from an indicator P-module.

e.g.

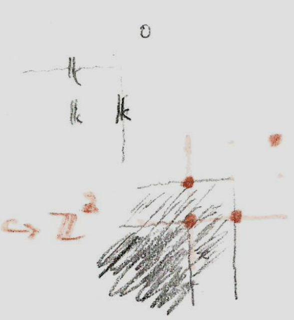


subordinate to encoding by  from before. (Why? fibers of π ...)

e.g. Thm f.d.!

a "see previous e.g."

$= \pi^*\left(0 \to \Bbbk[\ \cdot\ ] \longrightarrow \Bbbk[\ \cdot\ ] \to 0\right)$



$= \pi^* \iota^* \left(0 \to \Bbbk\left[\ \ {}^{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\right] \to \Bbbk\left[\ \ {}^{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}\right] \to 0\right)$ for $P \hookrightarrow \mathbb{Z}^2$

Syzygy Thm: Fix poset Q and Q-mod H with finite encoding $\pi: Q \to P$
$H \cong \pi^* M$.

1. H admits finite homological indicator res. subordinate to π.
2. " co "

Prop: Suppose $\iota: P \subseteq Z$. $\exists$ Z-mod $\iota_* M$ (pushforward) with
poset morphism
think $Z = \mathbb{Z}^m$

- $\iota_* M|_P = M$ and is universal with this property:
- $\iota_* M \to N$ whenever $N|_P = M$.

Pf: At $z \in Z$, $(\iota_* M)_z = \varinjlim_{p \preceq z} M_p$. "Kan extension" $\square$

Pf of Thm: $|P| < \infty \Rightarrow \iota: P \hookrightarrow \mathbb{Z}^m$ some m (order dim of P).

$N = \iota_* M$ f.d. because $|P| < \infty$: only finitely many $\varinjlim$ s, and they only change when some coord. of z passes that coord. of some $\iota(p)$.

N has finite flat or inj. res. by Thm f.d.

$\iota^*(\text{flat res } N) = \text{homological indicator res of } M$
inj. = co

$\pi^*(\text{res of } M) = \text{res of } H$ subordinate to π. $\square$

e.g. see previous e.g

Functoriality (see p. ⑥)

Prop: π^* is functorial: $M \to M'$
$\Rightarrow \pi^* M \to \pi^* M'$.

think A_i's and B_j's are downsets

On monomial matrices, if $M = \Bbbk[A_1] \oplus \dots \oplus \Bbbk[A_k] \longrightarrow \Bbbk[B_1] \oplus \dots \oplus \Bbbk[B_l] = M'$
or $M' \longleftarrow M$ if A_i's and B_j's are upsets

$$\pi^* \left(\begin{array}{c|c} & B_1 \cdots B_l \\ \begin{array}{c} A_1 \\ \vdots \\ A_k \end{array} & \begin{bmatrix} & & \\ & \varphi_{ij} & \\ & & \end{bmatrix} \end{array} \right) = \begin{array}{c|c} & \pi^{-1}(B_1) \cdots \pi^{-1}(B_l) \\ \begin{array}{c} \pi^{-1}(A_1) \\ \vdots \\ \pi^{-1}(A_k) \end{array} & \begin{bmatrix} & & \\ & \varphi_{ij} & \\ & & \end{bmatrix} \end{array}$$

or ...

Fringe presentation

Def: A <u>fringe presentation</u> of a $\mathcal{Q}$-mod H is a homomorphism

$F \to E$ with image $\cong H$, where

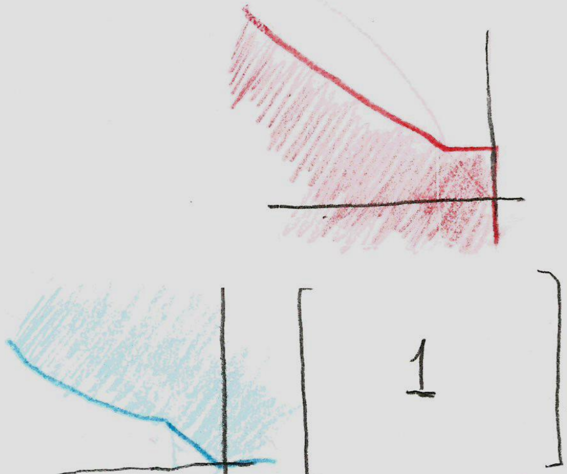
"fringe" = "frinj" = "free/injective" portmanteau

$$\begin{array}{c|c} & D_1 \cdots D_\ell \\ \hline \begin{matrix} U_1 \\ \vdots \\ U_k \end{matrix} & \left[\ \varphi_{ij}\ \right] \end{array}$$

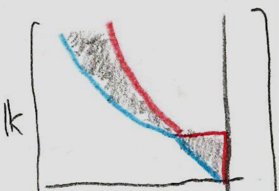
$$F = \Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k] \longrightarrow \Bbbk[D_1] \oplus \cdots \oplus \Bbbk[D_\ell] = E$$

<u>birth upsets</u>   <u>death downsets</u>

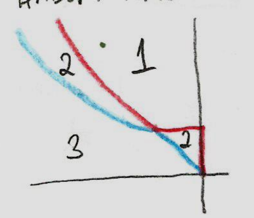
e.g.



fringe presentation of $\Bbbk$



fly wing module, essentially, though there it's really Hilbert function



<u>topological interpretation</u>: classes begin showing up at lower boundary of blue parameters
cease to appear at upper boundary of red parameters

<u>Cor</u>: Every finitely encoded $\mathcal{Q}$-module admits $\underbrace{\text{fringe presentation}}$.

<u>Pf</u>: Like 0^{th} map in Tate resolution: $\operatorname{coker}(F_1 \to F_0) = \ker(E^0 \to E^1)$. $\square$

Importance: interpretable data structure for statistics on sets of persistence modules

<u>Problem for stats</u> [Carlsson–Zomorodian 2008] (though lots of us here aren't at all surprised by this)

Even f.g. $\mathbb{Z}^n$-mods have moduli of dim ≥ 1: scalar entries φ_{ij} can vary continuously !

So how do you stats on sets of these?

<u>Answer 1</u>: geometric probability and stats on samples from stratified spaces — a completely different lecture series

- LLN
- CLT
- PCA ...

Answer 2: "no-moduli" conj. for $\underbrace{\text{fly wings}}$ $\qquad$ dim(fly wing) = 2

persistence in dim 0
codim 1
$\Downarrow$ this is all there is!

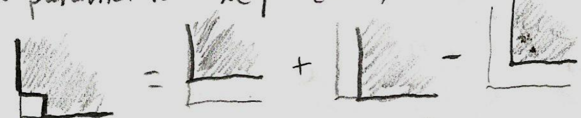
$\Rightarrow$ fixed $[\varphi_{ij}]$; only pay attention to

- moving U_i and D_i.
- $\Leftrightarrow$ rank function $\underbrace{(Q \times Q)_{\succeq \Delta}}_{\{(q,q') \mid q \preceq q'\}} \longrightarrow \mathbb{N}$

fringe presentation $\rightsquigarrow$ data structure for (semianalytic) functions on $(\mathbb{R}^n \times \mathbb{R}^n)_{\succeq \Delta}$ of this sort: level sets are semialgebraic (for fly wings and most applications)

- compute
- store
- manipulate / analyze statistically

Note: even for f.g. $\mathbb{Z}^n$-mods, fringe presentation is

- new
- more efficient than free presentation: topologically speaking, if a class appears for all $\mathbb{N}^2$-parameters except $\begin{bmatrix}0\\0\end{bmatrix}$, why remember 

Instead, ignore "obvious" relations/syzygies that collapse / gather arise from upsets/downsets not being (co)principal.

Exercise: $X =$ 

x-filtration by $\mathbb{N}$: 

y-filtration by $\mathbb{N}$: 

compute
- gens + rels
- fringe presentation

for filtration of X by $\mathbb{N}^2$.

Existence

You can and should ask me: your thms all assume existence of finite encoding;
When does that happen? Is existence restrictive?

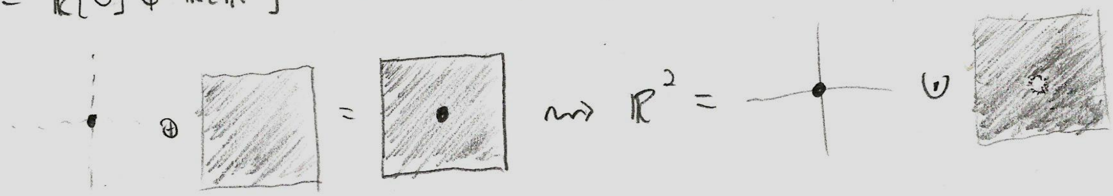
Def. Fix Q-mod. H.
H induces <u>isotypic regions</u>: classes for equivalence relation generated by

$a \sim b$ whenever
- $a \leq b$ in Q
- $H_a \xrightarrow{\sim} H_b$.

H is <u>tame</u> if
- $\dim_{\Bbbk} H_q < \infty \ \forall q \in Q$
- $\#$ isotypic regions $< \infty$

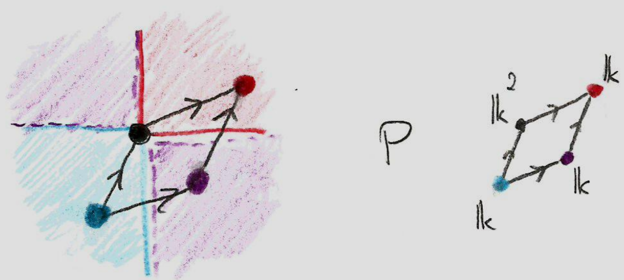
$\Big\}$ always happens in data analysis

e.g. $H = \Bbbk[0] \oplus \Bbbk[\mathbb{R}^2]$ induces isotypic regions $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$.

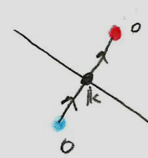


<u>Encoding Thm</u>: tame $\Rightarrow \exists$ finite encoding.

e.g. No poset map $\mathbb{R}^2 \to P$ has fibers $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$.

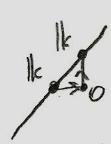


e.g. 1. $H = \Bbbk[\diagdown]$ over $\mathbb{R}^2$:



$a < b \Rightarrow H_a \to H_b$ is 0.
$\Rightarrow$ uncountably many isotypic regions, <u>but</u>: finite encodi[ng]

2. $H = \Bbbk[\diagup]$ over $\mathbb{R}^2$:



$\Rightarrow$ all maps $\Bbbk \nearrow \Bbbk$ are 0

$\Rightarrow$ uncountably many isotypic regions and no finite encoding

Conclusion: we think "finitely encoded" is the right notion. $\square$