Isotonian Algebras

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The 24th National School on Algebra
EMS Summer School on Multigraded Algebra and Applications
Moieciu de Jos, Romania, August 17-24, 2016

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- Gröbner basis of $J_{P,Q}$ (The defining ideal of $K[P, Q]$)
- Generators of $J_{P,Q}$

The following results are based on the joint work "Isotonian Algebras" with Mina Bigdeli, Takayuki Hibi, Jürgen Herzog and Akihiro Shikama.

Let $\mathcal{P}$ be the category of finite posets.

- Objects: finite posets
- Morphisms: isotone maps (i.e. order preserving maps)

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Let $\mathrm{Hom}(P, Q)$, the set of isotone maps from P to Q.

$\mathrm{Hom}(P, Q)$ is itself a poset with relation $\varphi_1 \leq \varphi_2$ if $\varphi_1(p) \leq \varphi_2(p)$ for all $p \in P$.

The K-algebra $K[P, Q]$

For any $\varphi \in \mathrm{Hom}(P, Q)$, we define the monomial

$$u_\varphi = \prod_{p \in P} x_{p,\varphi(p)}.$$

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If we let $Q = [2]$, then $K[P, [2]]$ is the classical Hibi ring with Krull dimension rank $P + 1$.

Also, it is well known that the Hibi ring is a normal Cohen–Macaulay domain.

Dimension of $K[P, Q]$

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Theorem. Let $P, Q \in \mathcal{P}$ and let r be the number of connected components of P and s be the number of connected components of Q. Then $\dim K[P, Q] = |P|(|Q| - s) + rs - r + 1$.

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- $K[P, [2]]$; Classical Hibi rings.
- $K[[2], Q]$; the edge ring of a bipartite graph.
- $K[P, [n]]$; the generalized Hibi ring (Latterplace algebras).

Normality of $K[P, Q]$

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Here $G(P)$ is the underlying graph of Hasse diagram of P.

Gröbner basis of $J_{P,Q}$

Theorem.
Let P be the chain and suppose that each connected component of Q is either rooted or a co-rooted. Then the defining ideal of $K[P, Q]$ admits a quadratic Gröbner basis and a squarefree initial ideal.

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Theorem: Let $R_1 = K[f_1, \dots, f_r]$ and $R_2 = K[g_1, \dots, g_s]$ be standard graded toric rings whose defining ideals are generated by squarefree binomials, then this is also the case for $R_1 * R_2$.

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Using the following

Lemma Let $P_1, P_2, \dots, P_r, Q \in \mathcal{P}$. Then

$$K[\sum_{i=1}^{r} P_i, Q] \cong K[P_1, Q] * K[P_2, Q] * \cdots * K[P_r, Q]$$

Generators of $J_{P,Q}$

Let $T = K[t_\varphi \ \varphi \in \mathrm{Hom}(P,Q)]$ and let $\phi : T \to K[P,Q]$ be the K-algebra homomorphism defined by $t_\varphi \mapsto u_\varphi = \prod_{p\in P} x_{p\varphi(p)}$. We denote the kernel of φ by $J_{P,Q}$.

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A typical generator of $J_{P,Q}$ of degree d looks like

$$f = t_{\varphi_1} \dots t_{\varphi_d} - t_{\psi_1} \dots t_{\psi_d}$$

with $\varphi_1, \dots, \varphi_d, \psi_1, \dots, \psi_d \in \mathrm{Hom}(P,Q)$.

We call the binomial $f = \prod_{k=1}^{d} t_{\varphi_i} - \prod_{k=1}^{d} t_{\varphi'_i}$ is special of type (p, π) with respect to $\varphi_1, \ldots, \varphi_d \in \mathrm{Hom}(P, Q)$, if

$$\varphi'_i(q) = \begin{cases} \varphi_{\pi(i)}(q), & \text{if } q \geq p, \\ \varphi_i(q), & \text{otherwise.} \end{cases}$$

Here π is a permutation on the set $\{1, \ldots, d\}$.

Theorem. Let P be a poset whose connected components are rooted or co-rooted poset. Then for any Q, the ideal I_{PQ} is generated by squarefree special binomials.

Quadratic generators

Let Q be a poset. A sequence $q_1, q_2, \ldots, q_{2m}$ of elements in Q is called a poset cycle of length $2m$ if

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We say that a poset cycle of length ≥ 6 has a chord if there exist an odd i and an even j with $j \ne i-1, i+1$ such that $q_i \le q_j$. (Here 0 is identified with $2m$).

A poset cycle is called *proper* if all elements q_i in the cycle are pairwise distinct and the element with even indexes are pairwise incomparable as well as elements with odd indexes.

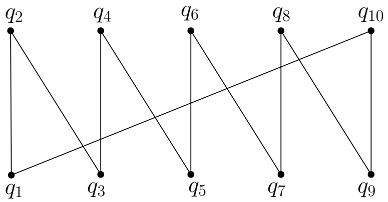

Figure: A proper poset cycle of length 10

Theorem. Let $P, Q \in \mathcal{P}$.

(a) If P is an antichain then $J_{P,Q}$ is generated by quadratic squarefree binomials.

(b) If P is not an antichain and each connected component of P is rooted or co-rooted, then the following conditions are equivalent:

(i) $J_{P,Q}$ is generated by quadratic binomials.

(ii) $J_{P,Q}$ is generated by quadratic squarefree binomials.

(iii) Each proper poset cycle of Q of length ≥ 6 has a chord.

Remark. If Q has a proper poset cycle of length $2d$ without a chord, then given any P, one can generate an irreducible binomial of degree d in $J_{P,Q}$.