On Polyomino Ideals

Sara Saeedi Madani
(Joint with: Jürgen Herzog)

Universität Osnabrück

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Let $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$.

We consider $(\mathbb{R}_+^2, \leq)$ as a partially ordered set with $(x, y) \leq (z, w)$ if $x \leq z$ and $y \leq w$.

Let $a, b \in \mathbb{N}^2$. Then the set $[a, b] = \{c \in \mathbb{N}^2 : a \leq c \leq b\}$ is called an interval.

Let $a = (i, j), b = (k, l) \in \mathbb{N}^2$ with $i < k$ and $j < l$.
Then the elements a and b are called diagonal corners, and the elements $c = (i, l)$ and $d = (k, j)$ are called anti-diagonal corners of $[a, b]$.

A cell C is an interval of the form $[a, b]$, where $b = a + (1, 1)$. The elements of C are called vertices of C. We denote the set of vertices of C by $V(C)$. The intervals $[a, a + (1, 0)]$, $[a + (1, 0), a + (1, 1)]$, $[a + (0, 1), a + (1, 1)]$ and $[a, a + (0, 1)]$ are called edges of C.

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Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. Then two cells C and D are called connected if there exists a sequence

$$\mathcal{C} : C = C_1, C_2, \dots, C_t = D$$

of cells of $\mathcal{P}$ such that for all $i = 1, \dots, t-1$ the cells C_i and C_{i+1} intersect in an edge.

If the cells in $\mathcal{C}$ are pairwise distinct, then $\mathcal{C}$ is called a path between C and D.

A finite collection of cells $\mathcal{P}$ is called a polyomino if every two cells of $\mathcal{P}$ are connected.

The vertex set of $\mathcal{P}$, denoted $V(\mathcal{P})$, is defined to be $\bigcup_{C\in\mathcal{P}} V(C)$.

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Polyominoes

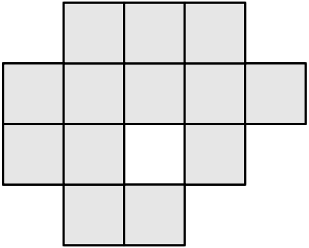
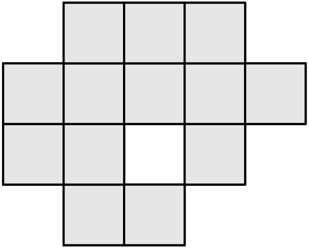


Figure: A polyomino

The name polyomino was invented by Solomon W. Golomb in 1953 and it was popularized by Martin Gardner.



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Polyomino Ideal

An inner interval I of a polyomino $\mathcal{P}$ is an interval with the property that all cells inside I belong to $\mathcal{P}$.

Let $\mathcal{P}$ be a polyomino and $S = K[x_a : a \in V(\mathcal{P})]$ be the polynomial ring with the indeterminates x_a over the field K. The 2-minor $x_a x_b - x_c x_d \in S$ is called an inner minor of $\mathcal{P}$ if $[a, b]$ is an inner interval of $\mathcal{P}$ with anti-diagonal corners c and d.

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Associated to $\mathcal{P}$ is the binomial ideal $I_{\mathcal{P}}$ in S, generated by all inner minors of $\mathcal{P}$. This ideal is called the polyomino ideal of $\mathcal{P}$, and the K-algebra $K[\mathcal{P}] = S/I_{\mathcal{P}}$ is called the coordinate ring of $\mathcal{P}$.

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Simple Polyominoes

Let $\mathcal{P}$ be a polyomino and $\mathcal{I}$ a rectangular polyomino such that $\mathcal{P} \subset \mathcal{I}$. Then the polyomino $\mathcal{P}$ is called simple, if each cell C which does not belong to $\mathcal{P}$ satisfies the following condition $(*)$:

there exists a path $\mathcal{C} : C = C_1, C_2, \ldots, C_t = D$ with $C_i \notin \mathcal{P}$ for all $i = 1, \ldots, t$ and such that D is not a cell of $\mathcal{I}$.

Simple Polyominoes

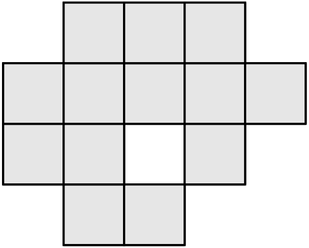

Figure: A polyomino which is not simple

Simple Polyominoes

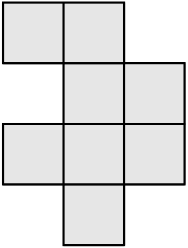

Figure: A simple polyomino

Let $\mathcal{P}$ be a polyomino and let $\mathcal{H}$ be the collection of cells $C \notin \mathcal{P}$ which do not satisfy condition $(*)$. The connected components of $\mathcal{H}$ are called the holes of $\mathcal{P}$.

Note that $\mathcal{P}$ is simple if and only if it is hole-free.

Simple Polyominoes

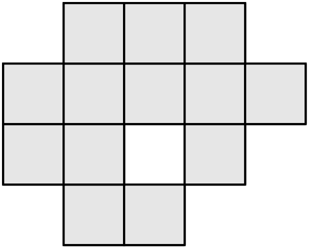

Figure: A polyomino which has a hole

Conjecture (Qureshi, 2012)

Let $\mathcal{P}$ be a simple polyomino. Then $I_{\mathcal{P}}$ is a prime ideal.

Admissible Labeling

For a polyomino $\mathcal{P}$, a function $\alpha : V(\mathcal{P}) \to \mathbb{Z}$ is called an admissible labeling of $\mathcal{P}$, if for all maximal horizontal and vertical edge intervals I of $\mathcal{P}$, we have

$$\sum_{a \in I} \alpha(a) = 0.$$

Admissible Labeling

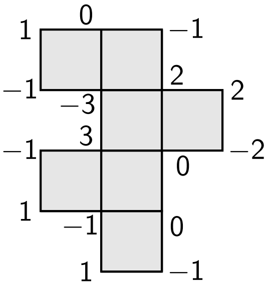

Figure: An admissible labeling

Let α be an admissible labeling of a polyomino $\mathcal{P}$. We may view α as a vector $\alpha \in \mathbb{Z}^n$, where n is the number of vertices of $\mathcal{P}$. By using this notation, we associate to α the binomial $f_\alpha = \mathbf{x}^{\alpha^+} - \mathbf{x}^{\alpha^-}$.

Let $J_\mathcal{P}$ be the ideal in S which is generated by the binomials f_α, where α is an admissible labeling of $\mathcal{P}$. By definition, it is clear that $I_\mathcal{P} \subset J_\mathcal{P}$.

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A polyomino $\mathcal{P}$ is called balanced if $f_\alpha \in I_\mathcal{P}$ for every admissible labeling α of $\mathcal{P}$.

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Balanced Polyominoes

Theorem (Herzog - Qureshi - Shikama, 2014)

Let $\mathcal{P}$ be a balanced polyomino. Then $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain of dimension $|V(\mathcal{P})| - |\mathcal{P}|$.

(Row or Column) Convex Polyominoes

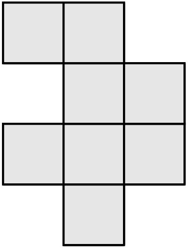

Figure: A row convex polyomino which is not column convex

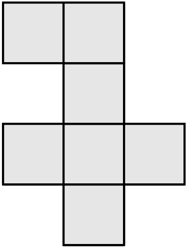

Figure: A tree-like polyomino

(Row or Column) Convex and Tree–like Polyominoes

Theorem (Herzog - Qureshi - Shikama, 2014)

Let $\mathcal{P}$ be a row or column convex, or a tree–like polyomino. Then $\mathcal{P}$ is balanced and simple.

(Row or Column) Convex and Tree–like Polyominoes

Corollary (Herzog - Qureshi - Shikama, 2014)

Let $\mathcal{P}$ be a row or column convex, or a tree-like polyomino. Then $K[\mathcal{P}]$ is a normal Cohen–Macaulay domain.

Simple = Balanced

Theorem (Herzog, -, 2015)

A polyomino is simple if and only if it is balanced.

Conjecture is proved!

Corollary (Herzog, -, 2015)

Let $\mathcal{P}$ be a simple polyomino. Then $K[\mathcal{P}]$ is a Cohen–Macaulay normal domain.

Even more!

Theorem (Qureshi - Shibuta - Shikama, 2015)

Let $\mathcal{P}$ be a simple polyomino. Then $K[\mathcal{P}]$ is a toric edge ring.

What else is prime?

- (Shikama, 2015) Rectangle minus rectangle.
- (Hibi-Qureshi, 2015) Rectangle minus convex.

Furthermore...

- Determining polyominoes $\mathcal{P}$ with exactly one hole where $I_{\mathcal{P}}$ is prime.
- Are they all radical?
- Studying the ideal of higher minors.
- Studying some algebraic invariants like regularity.

Thanks for your attention.