On the containment problem

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Definition

Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x_0, \ldots, x_n]$ be the ring of polynomials. For a homogeneous ideal $0 \neq I \subsetneq R$ its *m-th symbolic power* is

$$I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} \left(I^m R_P \cap R\right).$$

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Theorem (Zariski-Nagata)

Let $X \subset \mathbb{P}^n(\mathbb{K})$ be a projective variety (in particular reduced). Then $I(X)^{(m)}$ is generated by all forms which vanish along X to order at least m.

Let $Z = \{P_1, \ldots, P_s\}$ *be a finite set of points in* $\mathbb{P}^n(\mathbb{K})$. *Then*

$$I(Z) = I(P_1) \cap \ldots \cap I(P_s)$$

and

$$I(Z)^{(m)} = I(P_1)^m \cap \ldots \cap I(P_s)^m$$

for all $m \geq 1$.

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Theorem (Ein-Lazarsfeld-Smith, Hochster-Huneke)

If $m \geq \mathrm{bight}(I) r$, *then* $I^{(m)} \subset I^r$.

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Theorem (A simplified version)

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Can one improve the coefficient n in front of r?

Answer

No (Bocci, Harbourne).

Problem (Bocci, Harbourne, Huneke)

Does the containment

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hold for all r *and* $m \geq nr - (n-1)$*?*

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Let I be a saturated ideal of points in $\mathbb{P}^2(\mathbb{K})$. Is there the containment

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Problem (Harbourne, Huneke)

Let $M = < x_0, \ldots, x_n >$. Does the containment

$$I^{(m)} \subset M^{r(n-1)} I^r$$

hold for $m \geq nr$?

Theorem

The containment

$$I^{(nr-(n-1))} \subset I^r$$

holds for

a) *arbitrary ideals in characteristic* 2*;*

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d) *ideals of general points in* $\mathbb{P}^2$ *and* $\mathbb{P}^3$*.*

Theorem (Seceleanu)

Let $I \subset R$ be a homogeneous ideal. There is an associated exact sequence

$$0 \to I^r/I^m \to R/I^m \xrightarrow{\pi} R/I^r \to 0.$$

The following conditions are equivalent:

i) *there is the containment $I^{(m)} \subset I^r$,*

ii) *the induced map $H^0_M(\pi) : H^0_M(R/I^m) \to H^0_M(R/I^r)$ is the zero map.*

Theorem (Dumnicki, Sz., Tutaj-Gasińska)

The containment

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fails for the ideal I of points

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:1), & P_5 = (1:\varepsilon:\varepsilon^2), & P_6 = (1:\varepsilon^2:\varepsilon), \\ P_7 = (\varepsilon:1:1), & P_8 = (1:\varepsilon:1), & P_9 = (1:1:\varepsilon), \\ P_{10} = (\varepsilon^2:1:1), & P_{11} = (1:\varepsilon^2:1), & P_{12} = (1:1:\varepsilon^2). \end{array}$$

in $\mathbb{P}^2(\mathbb{C})$.

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in $\mathbb{P}^2(\mathbb{C})$.

Remark

These are all intersection points of the dual Hesse configuration of lines.

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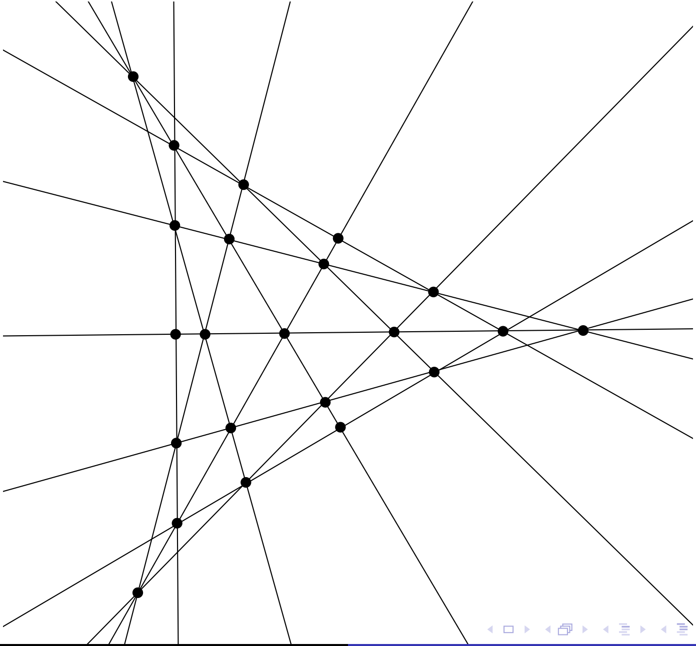
No counterexamples in $\mathbb{P}^n$ *for* $n \geq 3$

Boröczky configuration of 12 lines



Example

Let $\mathbb{K}$ be a field of odd characteristic p and let $\mathbb{L}$ be its subfield of order p. Let $N = \frac{p+1}{2}$ and let Z be the set of all but one $\mathbb{L}$-points in $\mathbb{P}^N(\mathbb{K})$. Then for the ideal $I = I(Z)$ there is

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Example

Let the numbers p and N be so that $p \equiv 1 \pmod N$ and $p > (N-1)^2$. Let Z be the set of all but one $\mathbb{L}$-points in $\mathbb{P}^N(\mathbb{K})$. Then for $r = \frac{p-1}{N} + 1$ there is

$$I^{(p)} \not\subseteq I^r.$$

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- *Cooper, Embree, Ha and Hoefel: Symbolic powers of monomial ideals.*

Definition

For a graded ideal I its *initial degree* $\alpha(I)$ is the least number t such that $I_t \neq 0$.
The *Waldschmidt constant* of I is the real number

$$\widehat{\alpha}(I) = \inf_{m \geq 1} \frac{\alpha(I^{(m)})}{m}.$$

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Conjecture (Chudnovsky)

Let I be a saturated ideal of points in $\mathbb{P}(\mathbb{K})$. Then

$$\widehat{\alpha}(I) \geq \frac{\alpha(I) + n - 1}{n}.$$

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Conjecture (Nagata)

Let I be a saturated ideal of $s \geq 10$ very general points in $\mathbb{P}(\mathbb{C})$. Then

$$\alpha(I^{(m)}) > m\sqrt{s}.$$

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Let S be a smooth complex surface. Then there is a number b such that

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Remark

Negativity on blow ups of $\mathbb{P}^2(\mathbb{C})$ gets worst (in terms of Harbourne constants) for intersection points of configurations of lines with no simple intersection points.

Last but not least