Toric ideals

Apostolos Thoma

Department of Mathematics
University of Ioannina

EMS Summer School on Multigraded Algebra and Applications
Moieciu, Romania

Toric ideals

Let $A = \{\mathrm{a}_1, \dots, \mathrm{a}_n\} \subseteq \mathbb{Z}^m$ be a vector configuration in $\mathbb{Q}^m$ and $\mathbb{N}A := \{l_1\mathrm{a}_1 + \cdots + l_n\mathrm{a}_n \mid l_i \in \mathbb{N}_0\}$ the corresponding affine semigroup. Let $A = [\mathrm{a}_1 \dots \mathrm{a}_n] \in \mathbb{Z}^{m\times n}$ be an integer matrix with columns a_i. For a vector $\mathrm{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ we let u^+, u^- be the unique vectors in $\mathbb{N}^n$ with disjoint support such that $\mathrm{u} = \mathrm{u}^+ - \mathrm{u}^-$.

Definition

The toric ideal I_A of A is the ideal in $K[x_1, \cdots, x_n]$ generated by all binomials of the form $x^{\mathrm{u}^+} - x^{\mathrm{u}^-}$ where $\mathrm{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$.

A toric ideal is a binomial ideal.

Example

Let

$$A = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

Then $\begin{pmatrix} 5 \\ -4 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ belongs to the $\mathrm{Ker}_{\mathbb{Z}}(A)$ since

$$A = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Toric ideals

Example

For the vector $u = \begin{pmatrix} 5 \\ -4 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ we have $u^+ = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $u^- = \begin{pmatrix} 0 \\ 4 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Therefore the binomial $x^{\mathrm{u}^+} - x^{\mathrm{u}^-} = x_1^5 x_5 x_6 - x_2^4 x_3^3 \in I_A$.

Toric ideals

Let $A = \{\mathrm{a}_1, \ldots, \mathrm{a}_n\} \subset \mathbb{Z}^m$ be a vector configuration in $\mathbb{Q}^m$. Let K be any field. We grade the polynomial ring $K[x_1, \ldots, x_m]$ by setting $\deg_A(x_i) = \mathrm{a}_i$ for $i = 1, \ldots, m$. The *A-degree* of the monomial $\mathrm{x}^{\mathrm{u}} := x_1^{u_1} \cdots x_m^{u_m}$ is defined to be

$$\deg_A(\mathrm{x}^{\mathrm{u}}) := u_1\mathrm{a}_1 + \cdots + u_m\mathrm{a}_m \in \mathbb{N}A,$$

where $\mathrm{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$.

Definition

The *toric ideal* I_A associated to A is the ideal generated by all the binomials $\mathrm{x}^{\mathrm{u}} - \mathrm{x}^{\mathrm{v}}$ such that $\deg_A(\mathrm{x}^{\mathrm{u}}) = \deg_A(\mathrm{x}^{\mathrm{v}})$.

For such binomials, we define $\deg_A(\mathrm{x}^{\mathrm{u}} - \mathrm{x}^{\mathrm{v}}) := \deg_A(\mathrm{x}^{\mathrm{u}})$.

Toric ideals

Example

The A-degree of the binomial $x_1x_6 - x_2x_4$ is

$$\deg_A(x_1x_6) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \deg_A(x_2x_4)$$

I_A is minimally generated by:
$\{x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_4^2x_5 - x_3x_6^2, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6,$
$x_1x_4^2 - x_3^2x_6, x_2^2x_3 - x_1^2x_5, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2\}$.
The A-degrees of the binomials are accordingly

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix},$$

$$\begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

Toric varieties

Toric ideals are the defining ideals of toric varieties.

$$V(I_A) = \{P \in K^n | f(P) = 0 \text{ for every } f \in I_A\}$$

It is the Zariski closure of the set of points

$$(t^{a_1}, t^{a_2}, \cdots, t^{a_n})$$

where $t \in (K - \{0\})^m$ and $a_1, \cdots, a_n$ are the columns of the matrix A.

Example

If A is a row matrix, $[m_1, m_2, \cdots, m_n]$, then the toric variety is a monomial curve in K^n: the set of all points in the form $(t^{m_1}, t^{m_2}, \cdots, t^{m_n})$ where $t \in K$.

Toric varieties

Let

$$A = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

Then the toric variety $V(I_A)$ is the Zariski closure of the set of points

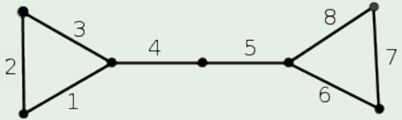
$$(t_1^2 t_2, t_1 t_2^2, t_1^2 t_3, t_1 t_3^2, t_2^2 t_3, t_2 t_3^2)$$

where $t = (t_1, t_2, t_3) \in (K - \{0\})^3$.

Graphs

A simple graph G consists of a set of vertices $V(G) = \{v_1, \dots, v_m\}$ and a set of edges $E(G) = \{e_1, \dots, e_n\}$, where an edge $e \in E(G)$ is an unordered pair of vertices, $\{v_i, v_j\}$. Let A_G be the vertex-edge incident matrix of the graph G. This is am $m \times n$ matrix with $0/1$ entries. The rows are indexed by the vertices and the columns by the edges. The element in the ij position of the matrix A_G is 1 if the vertex v_i belongs to the edge e_j, otherwise is zero.

Example



The vertex-edge incidence matrix of G.

$$A_G = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Toric ideals of graphs

With I_G we denote the toric ideal I_{A_G} in $\mathbb{K}[e_1, \dots, e_n]$, where A_G is the vertex-edge incidence matrix of G.

Let a_e be the column of A_G which corresponds to the edge e. Then the $deg_A(e) = a_e$, which is an m-column that has all the elements zero except two 1.
But one can associate with an edge $e = \{v_s, v_t\} \in E(G)$ the element $v_s + v_t$ in the free abelian group $\mathbb{Z}^n$ with basis the set of vertices of G and may think that $deg_A(e) = v_s + v_t$.

Graphs

Definition

- A walk connecting $v_{i_1} \in V(G)$ and $v_{i_{q+1}} \in V(G)$ is a finite sequence of the form

$$w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$$

  with each $e_{i_j} = \{v_{i_j}, v_{i_{j+1}}\} \in E(G)$.
- Length of the walk w is called the number q of edges of the walk.
- An even walk is a walk of even length.
- An odd walk is a walk of odd length.

Graphs

Definition

A walk $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$ is called closed if $v_{i_{q+1}} = v_{i_1}$.
A cycle is a closed walk

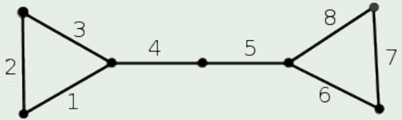
$$(\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_1}\})$$

with $v_{i_k} \neq v_{i_j}$, for every $1 \leq k < j \leq q$.

Note that, although the graph G has no multiple edges, the same edge e may appear more than once in a walk. In this case e is called multiple edge of the walk w.

Toric ideals of Graphs

Example



- (e_1, e_2, e_3) is closed odd walk, actualy is a cycle.
- $(e_1, , e_2, e_3, e_1, e_2, e_3)$ is a closed even walk. All of the edges are double edges of the walk.
- $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_5, e_4)$ is a closed even walk. The edges e_4, e_5 are double edges of the walk.

Toric ideals of Graphs

Given an even closed walk

$$w = (e_{i_1}, e_{i_2}, \dots, e_{i_{2q}}) = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_q}, v_{i_1}\})$$

of the graph G we denote by

$$E^+(w) = \prod_{k=1}^{q} e_{i_{2k-1}}, \ E^-(w) = \prod_{k=1}^{q} e_{i_{2k}}$$

and by B_w the binomial

$$B_w = \prod_{k=1}^{q} e_{i_{2k-1}} - \prod_{k=1}^{q} e_{i_{2k}}.$$

Note that

$$deg_A(E^+(w)) = deg_A(\prod_{k=1}^{q} e_{i_{2k-1}}) = (v_{i_1}+v_{i_2})+(v_{i_3}+v_{i_4})+\cdots+(v_{i_{q-1}}+v_{i_q}) =$$

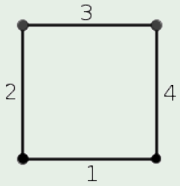
$$= (v_{i_2}+v_{i_3})+(v_{i_4}+v_{i_5})+\cdots+(v_q+v_{i_1}) = deg_A(\prod_{k=1}^{q} e_{i_{2k}}) = deg_A(E^-(w))$$

Therefore B_w belongs to the toric ideal I_G.

Toric ideals of graphs

Example

Let G be the following graph with 4 vertices and 4 edges.



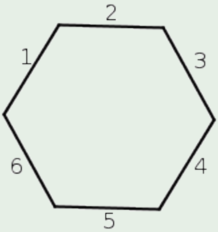
Then

$$A_G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

For the even closed walk $w = (e_1, e_2, e_3, e_4)$ we have $E^+(w) = e_1e_3$, $E^-(w) = e_2e_4$ and $B_w = e_1e_3 - e_2e_4$. In fact the toric ideal associated with A_G is $I_G = < e_1e_3 - e_2e_4 >$.

Toric ideals of Graphs

Example

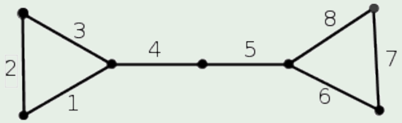


For the even closed walk $w = (e_1, e_2, e_3, e_4, e_5, e_6)$ we have that $E^+(w) = e_1 e_3 e_5$ and $E^-(w) = e_2 e_4 e_6$ therefore

$$B_w = e_1 e_3 e_5 - e_2 e_4 e_6.$$

Note that $deg_G(e_1 e_3 e_5) = deg_G(e_2 e_4 e_6) = v_1 + v_2 + v_3 + v_4 + v_5 + v_6$.

Example

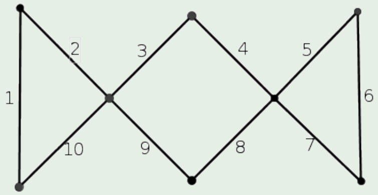


For the even closed walk $w = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_5, e_4)$ we have that $E^+(w) = e_1 e_3 e_5 e_7 e_5$ and $E^-(w) = e_2 e_4 e_6 e_8 e_4$ therefore

$$B_w = e_1 e_3 e_5^2 e_7 - e_2 e_4^2 e_6 e_8.$$

Note that
$deg_G(e_1 e_3 e_5^2 e_7) = deg_G(e_2 e_4^2 e_6 e_8) = v_1 + v_2 + 2v_3 + 2v_4 + 2v_5 + v_6 + v_7$.

Example

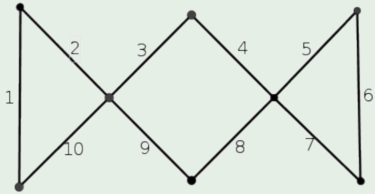


Note that different walks may correspond to the same binomial. For example both walks $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10})$ and $(e_1, e_2, e_9, e_8, e_5, e_6, e_7, e_4, e_3, e_{10})$ correspond to the same binomial

$$B_w = e_1 e_3 e_5 e_7 e_9 - e_2 e_4 e_6 e_8 e_{10}.$$

Toric ideals of Graphs

Example



Also note that for certain even closed walks w the binomial B_w may be zero, for example take w to be the even closed walk $(e_1, e_2, e_9, e_8, e_5, e_5, e_8, e_9, e_2, e_1)$ we have

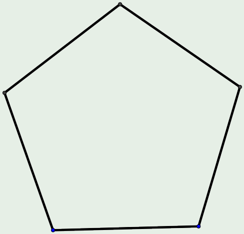
$$B_w = e_1 e_9 e_5 e_8 e_2 - e_2 e_8 e_5 e_9 e_1 = 0.$$

For the walk $\xi = (e_1, e_2, e_{10}, e_1, e_2, e_{10})$ we have

$$B_\xi = e_1 e_{10} e_2 - e_2 e_1 e_{10} = 0.$$

Toric ideals of Graphs

Example



There are examples that for every even closed walk w the binomial B_w is zero, in these cases

$$I_G = 0.$$

Toric ideals of Graphs

Theorem (R. Villarreal)

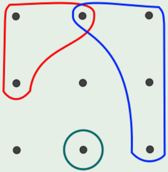
The toric ideal I_G of a graph G is generated by binomials of the form B_w, where w is an even closed walk.

Hypergraphs

A (multi)hypergraph H consists of a set of vertices $V(H) = \{v_1, \ldots, v_m\}$ and a set of edges $E(H) = \{E_1, \ldots, E_n\}$, where an edge $E \in E(H)$ is a subset of the vertices. Let A_H be the vertex-edge incident matrix of the graph G. This is am $m \times n$ matrix with $0/1$ entries. The rows are indexed by the vertices and the columns by the edges. The element in the ij position of the matrix A_H is 1 if the vertex v_i belongs to the edge E_j, otherwise is zero.

Any $m \times n$ matrix with $0/1$ entries and nonzero columns give rise to a (multi)hypergraph.

Example



The vertex-edge incidence matrix of H.

$$A_G = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Toric ideals of hypergraphs

Definition

Let (E_{blue}, E_{red}) be a multiset collection of edges of $H = (V, E)$. We denote by $\deg_{blue}(v)$ and $\deg_{red}(v)$ the number of edges of E_{blue} and E_{red} containing the vertex v, respectively. We say that (E_{blue}, E_{red}) are balanced on V if $\deg_{blue}(v) = \deg_{red}(v)$ for each vertex $v \in V$. If (E_{blue}, E_{red}) are balanced on V then we say that (E_{blue}, E_{red}) is a monomial walk.

Every monomial walk encodes a binomial

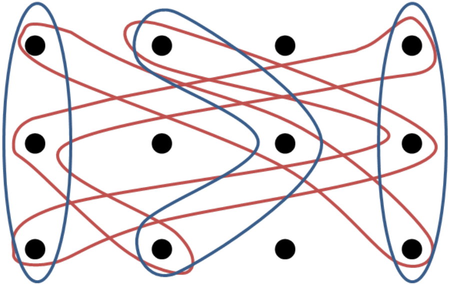
$$f_{E_{blue}, E_{red}} = \prod_{E \in E_{blue}} E - \prod_{E \in E_{red}} E$$

in I_H.

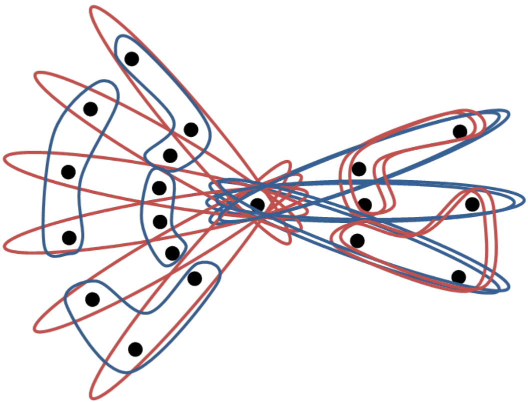
Theorem (Petrovic, Stassi)

The toric ideal I_H of a hypergraph is generated by binomials corresponding to monomial walks.

Toric ideals of hypergraphs



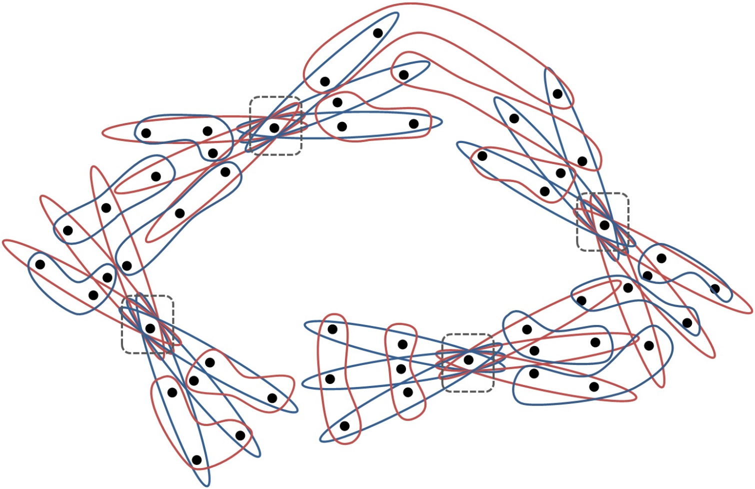
$$f_{E_{blue},E_{red}} = \prod_{E \in E_{blue}} E - \prod_{E \in E_{red}} E = E_1 E_2 E_3 - E_4 E_5 E_6$$



$$f_{E_{blue},E_{red}} = \prod_{E \in E_{blue}} E - \prod_{E \in E_{red}} E =$$

$$E_1 E_2 E_3 E_4 E_5^2 E_6^2 E_7^2 - E_8 E_9 E_{10} E_{11} E_{12} E_{13} E_{14}^2 E_{15}^2$$

Toric ideals of hypergraphs



Binomials in a toric ideal

Toric ideals are binomial ideals.
There are certain sets of binomials that are important:

- Graver basis
- Circuits
- Markov bases
- Indispensable binomials
- reduced Gröbner basis
- universal Gröbner basis

Graver basis

Definition

An irreducible binomial $x^{\mathrm{u}} - x^{\mathrm{v}}$ in I_A is called *primitive* if there exists no other binomial $x^{\mathrm{a}} - x^{\mathrm{b}} \in I_A$ such that x^{a} divides x^{u} and x^{b} divides x^{v}.

Definition

The set of all primitive binomials of a toric ideal I_A is called the Graver basis of I_A.

Graver basis

Let $A = [3\ 4\ 5]$ then the binomial $x_1^3 x_2^4 - x_3^5$ belongs to the toric ideal I_A and is not primitive, since the binomial $x_1^2 x_2 - x_3^2 \in I_A$ and

$$x_1^2 x_2 | x_1^3 x_2^4,$$

$$x_3^2 | x_3^5.$$

In this example there are 7 primitive binomials :
$x_1^4 - x_2^3, x_1 x_3 - x_2^2, x_1^3 - x_2 x_3, x_1^2 x_2 - x_3^2, x_1^5 - x_3^3, x_1 x_2^3 - x + 3^3, x_2^5 - x_3^4$.

Conformal sum

Definition

Let $u, w_1, w_2 \in \mathrm{Ker}_{\mathbb{Z}}(A)$ be such that $u = w_1 + w_2$. We say that the above sum is a conformal decomposition of u and write $u = w_1 +_c w_2$ if

$$u^+ = w_1^+ +_c w_2^+ \text{ and } u^- = w_1^- +_c w_2^- .$$

If both w_1 and w_2 are non-zero, we call such a decomposition proper.

Note that the above condition means that:

- if the i-coordinate of u is positive then the i-coordinates of w_1, w_2 are positive or zero
- if the i-coordinate of u is negative then the i-coordinates of w_1, w_2 are negative or zero
- if the i-coordinate of u is zero then both the i-coordinates of w_1, w_2 are zero.

Graver basis

Definition

The Graver basis of A, consists of the nonzero vectors in $\mathrm{Ker}_{\mathbb{Z}}(A)$ for which there is no proper conformal decomposition.

The Graver basis of A consists of vectors in $\mathrm{Ker}_{\mathbb{Z}}(A)$ and the Graver basis of I_A consists of binomials in I_A. Note also that if $u = w_1 +_c w_2$ then $-u = (-w_1) +_c (-w_2)$. Therefore if u belongs to the Graver basis of A then $-u$ belongs to the Graver basis of A.

The binomial $x^{u^+} - x^{u^-}$ is in the Graver basis of the toric ideal I_A if and only if the vector u is the Graver basis of A.

Theorem

The Graver basis is a finite set.

Graver basis

Every element v in $\mathrm{Ker}_{\mathbb{Z}}(A)$ can be written as a conformal sum of elements in the Graver basis of A.

$$v = u_1 +_c u_2 +_c \cdots +_c u_s$$

Where $u_1, u_2, \cdots, u_s$ are not necessarily different and belong in the Graver basis of A and conformal means $v = u_1 + u_2 + \cdots + u_s$ and

- if the i-coordinate of v is positive then the i-coordinates of all u_j are positive or zero
- if the i-coordinate of v is negative then the i-coordinates of u_j are negative or zero
- if the i-coordinate of v is zero then all the i-coordinates of w_j are zero.

Graver basis

Let $A = [3\ 4\ 5]$ then the Graver basis of A consists of the following 7 elements:
$(4, -3, 0), (1, -2, 1), (3, -1, -1), (2, 1, -2), (5, 0, -3), (1, 3, -3), (0, 5, -4)$.
The element $(3, 4, -5)$ belongs to the kernel of A and can be written as a conformal sum:

$$(3, 4, -5) = (2, 1, -2) +_c (1, 3, -3).$$

Note also that

$$(3, 4, -5) = (3, -1, -1) + (0, 5, -4)$$

but this sum is not conformal.

Circuits

Definition

A non-zero vector $u \in \mathrm{Ker}_{\mathbb{Z}}(A)$ is called a circuit of A if its support

$$\mathrm{supp}(u) = \{i | u_i \neq 0\}$$

is minimal with respect to inclusion and the coordinates of u are relatively prime.

Definition

An irreducible binomial in I_A is called a circuit of I_A if its support

$$\mathrm{supp}(u) = \{x_i | u_i \neq 0\}$$

is minimal with respect to inclusion.

If $u = (u_1, u_2, \cdots, u_n)$ is a circuit then the vectors $\{a_i | i \in \text{supp}(u)\}$ are linearly dependent but any subset of them are linearly independent.

Let A be a $d \times n$ matrix of rank d and let $u = (u_1, u_2, \cdots, u_n)$ be a circuit. Let $\text{supp}(u) = \{i_1, \cdots, i_r\}$ then the $d \times r$-matrix $[a_{i_1}, a_{i_2}, \cdots, a_{i_r}]$ has rank $r - 1$. The vectors $a_{i_1}, a_{i_2}, \cdots, a_{i_{r-1}}$ are linearly independent therefore we can find vectors $a_{i_{r+1}}, a_{i_{r+2}}, \cdots, a_{i_{d+1}}$ such that $a_{i_1}, a_{i_2}, \cdots, a_{i_{r-1}}, a_{i_{r+1}}, a_{i_{r+2}}, \cdots, a_{i_{d+1}}$ is a basis for the column space of A. Then the $d \times (d+1)$-matrix $[a_{i_1}, a_{i_2}, \cdots, a_{i_{d+1}}]$ has rank d. The kernel of this matrix is generated by

$$\sum_{j=1}^{d+1} (-1)^j \det(a_{i_1}, a_{i_2}, \cdots, a_{i_{j-1}}, a_{i_{j+1}}, \cdots, a_{i_{d+1}}) e_{i_j},$$

where e_{i_j} is the i_j-unit vector. Since this is an integer vector and u is a circuit it must be an integer multiple of u. Therefore $u_{i_j} = 1/g((-1)^j \det(a_{i_1}, a_{i_2}, \cdots, a_{i_{j-1}}, a_{i_{j+1}}, \cdots, a_{i_{d+1}}) e_{i_j})$, where $g = gcd(\det(a_{i_1}, a_{i_2}, \cdots, a_{i_{j-1}}, a_{i_{j+1}}, \cdots, a_{i_{d+1}}))|1 \leq j \leq r)$.

Circuits

Theorem

For every element v in $\mathrm{Ker}_{\mathbb{Z}}(A)$ there exist an integer multiple of it that can be written as a conformal sum of circuits of A.

$$kv = c_1 +_c c_2 +_c \cdots +_c c_s$$

This theorem implies also that for every element v in $\mathrm{Ker}_{\mathbb{Z}}(A)$ there exist a circuit c such that $supp(c^+) \subset supp(u^+)$ and $supp(c^-) \subset supp(u^-)$.

Example

Let $A = [3\ 4\ 5]$ then the Circuits of A are the following 3 elements: $(4, -3, 0), (5, 0, -3), (0, 5, -4)$. The element $(3, 4, -5)$ belongs to the kernel of A and a multiple of it can be written as a conformal sum of circuits:

$$5(3, 4, -5) = 3(5, 0, -3) +_c 4(0, 5, -4).$$

Circuits

Theorem

Let I_A be a toric ideal and C_A the ideal generated by the circuits then $I_A = rad(C_A)$.

Theorem

Let I_A be a toric ideal and C_A the ideal generated by the circuits then $V(I_A) = V(C_A)$.

Markov basis

Definition

A Markov basis of A is a finite subset M of $\mathrm{Ker}_{\mathbb{Z}}(A)$ such that whenever $\mathrm{w}, \mathrm{u} \in \mathbb{N}^n$ and $\mathrm{w} - \mathrm{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ (i.e. $A\mathrm{w}^t = A\mathrm{u}^t$), there exists a subset $\{\mathrm{v}_i : i = 1, \ldots, s\}$ of M that connects w to u. This means that $(\mathrm{w} - \sum_{i=1}^{p} \mathrm{v}_i) \in \mathbb{N}^n$ for all $1 \leq p \leq s$ and $\mathrm{w} - \mathrm{u} = \sum_{i=1}^{s} \mathrm{v}_i$. A Markov basis M of A is minimal if no subset of M is a Markov basis of A.

Note that the $deg_A(x^w) = Aw^t$ therefore x^w and x^u have the same A-degree. The set of all elements that have the same degree as x^w is called the fiber of x^w and is denoted:

$$deg^{-1}(x^w).$$

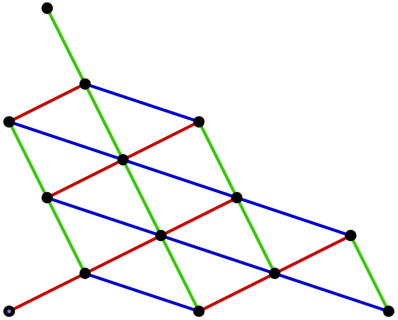
The elements $\mathrm{v}_i \in M$ are elements in $\mathrm{Ker}_{\mathbb{Z}}(A)$ therefore $A\mathrm{v}_i^t = 0$ which means that

$$x^{(\mathrm{w} - \sum_{i=1}^{p} \mathrm{v}_i)} \in deg^{-1}(x^w).$$

Therefore a Markov basis is a set of moves that connects any two elements of the same fiber by moving inside the fiber.

Markov basis

Let $A = [3\ 4\ 5]$ and I_A the corresponding toric ideal. A minimal Markov basis for I_A is $(-3, 1, 1)$, $(1, -2, 1)$, $(2, 1, -2)$. The fiber of all the monomial having A-degree 35 consists of 14 elements:
$x_1^{10}x_3, x_1^9x_2^2, x_1^7x_2x_3^2, x_1^6x_2^3x_3,$
$x_1^5x_2^5, x_1^5x_3^4, x_1^4x_2^2x_3^3, x_1^3x_2^4x_3^2, x_1^2x_2^6x_3, x_1^2x_2x_3^5, x_1x_2^8, x_1x_2^3x_3^5, x_2^5x_3^3, x_3^7$.



Markov basis

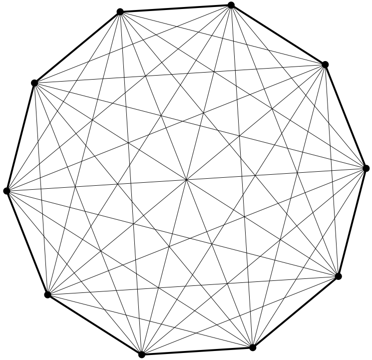
Theorem (Diaconis-Sturmfels 1998)

M is a minimal Markov basis of A if and only if the set $\{x^{\mathrm{u}^+} - x^{\mathrm{u}^-} : \mathrm{u} \in M\}$ is a minimal generating set of I_A.

Definition

We call a minimal Markov basis of I_A any minimal generating set of I_A.

Markov bases



In the toric ideal of the complete graph on 10 vertices there are

$$3^{210}$$

different minimal Markov bases. Every minimal Markov basis contains 420 elements.

Toric ideals

Apostolos Thoma

Department of Mathematics
University of Ioannina

EMS Summer School on Multigraded Algebra and Applications
Moieciu, Romania