Toric ideals and Toric ideals of graphs

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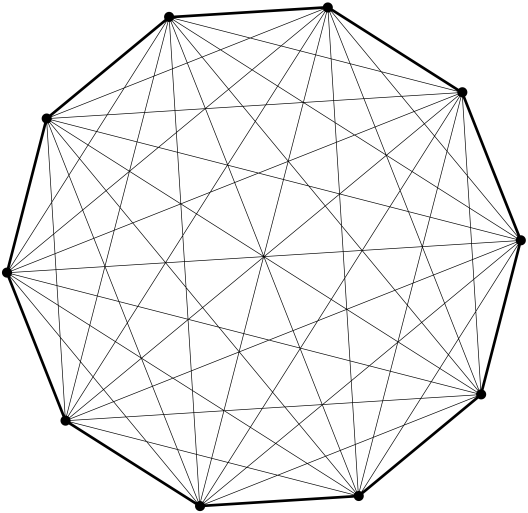
Binomials in a toric ideal

- Graver basis
- Circuits
- minimal Markov bases
- indispensables
- reduced Gröbner bases
- universal Gröbner basis



The complete graph on 10 vertices.

Graver basis

Toric ideal of the complete graph with 10 vertices.

- Every Markov basis contains 420 elements.
- There are 3^{210} different minimal Markov bases.
- There are $4,866,750$ circuits.
- There are $13,825,350$ binomials in the Universal Gröbner basis.
- There are $14,127,750$ binomials in the Graver basis.

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Universal Gröbner bases

Theorem (B. Sturmfels)

For any toric ideal I_A we have that the Universal Gröbner basis is a subset of the Graver basis.

Universal Gröbner bases

Theorem (B. Sturmfels)

For any toric ideal I_A we have that the set of circuits is a subset of the Universal Gröbner basis .

Circuits and reduced Gröbner basis of toric ideals

Definition

An irreducible binomial $x^{u^+} - x^{u^-}$ in I_A is called a circuit of I_A if its support

$$\text{supp}(x^{u^+} - x^{u^-}) = \{x_i | u_i \neq 0\}$$

is minimal with respect to inclusion.

Theorem

Let $<$ be a monomial order on $k[x_1, \dots, x_n]$ and I_A a toric ideal. Then $\{x^{{u_1}^+} - x^{{u_1}^-}, x^{{u_2}^+} - x^{{u_2}^-}, \cdots, x^{{u_s}^+} - x^{{u_s}^-}\}$ with $x^{{u_1}^+} > x^{{u_1}^-}, x^{{u_2}^+} > x^{{u_2}^-}, \cdots, x^{{u_s}^+} > x^{{u_s}^-}$ is the reduced Gröbner basis with respect to the monomial order $<$ if and only if $x^{{u_1}^+}, x^{{u_2}^+}, \cdots, x^{{u_s}^+}$ are the minimal monomial generators of $in_<(I_A)$ and $x^{{u_1}^-}, x^{{u_2}^-}, \cdots, x^{{u_s}^-}$ are standard monomials.

Proof.

Let $x^{u^+} - x^{u^-}$ be a circuit. We will prove that there exists an appropriate monomial order $<$ such that $x^{u^+} - x^{u^-}$ belongs to the reduced Gröbner basis with respect to the monomial order $<$.
Take any elimination order with the variables not belonging to the support of $x^{u^+} - x^{u^-}$ greater than the variables belonging to the support of $x^{u^+} - x^{u^-}$.
We claim that $x^{u^+} - x^{u^-}$ belongs to the reduced Gröbner basis with respect to this monomial order $<$.
Suppose not. Then either the initial monomial of $x^{u^+} - x^{u^-}$ is not a minimal generator of $In_<(I_A)$ or the other monomial is not standard. In both cases there exists $x^{v^+} - x^{v^-}$ in the reduced Gröbner basis of the toric ideal I_A, with $x^{v^+} > x^{v^-}$ such that x^{v^+} divides a monomial of $x^{u^+} - x^{u^-}$. W.l.o.g. we can assume that $x^{v^+}|x^{u^+}$. Then $\text{supp}(x^{v^+}) \subset \text{supp}(x^{u^+}) \subset \text{supp}(x^{u^+} - x^{u^-})$.
From the choice of the monomial order we see that also $\text{supp}(x^{v^-}) \subset \text{supp}(x^{u^+} - x^{u^-})$. But $x^{u^+} - x^{u^-}$ is a circuit and both $x^{u^+} - x^{u^-}$ and $x^{v^+} - x^{v^-}$ are irreducible, therefore $x^{u^+} - x^{u^-} = x^{v^+} - x^{v^-}$, a contradiction. □

Universal Gröbner bases

Theorem

For any toric ideal I_A we have $Circuits_A \subset UGB_A \subset Graver_A$.

What about the minimal Markov bases and the indispensables?
(generic, $I_{K_{10}}$)

Unimodular toric ideals

Definition

Let $A \in \mathbb{Z}^{m \times n}$ be a matrix of rank d. Then A is called unimodular if all non-zero $d \times d$-minors of A have the same absolute value. The toric ideal I_A is called unimodular if the matrix A is unimodular.

Unimodular toric ideals

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Unimodular toric ideals

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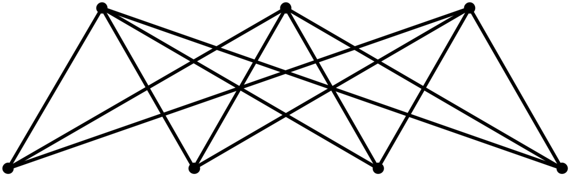


Unimodular toric ideals

Theorem

For a unimodular toric ideal the set of circuits is the Graver basis of I_A.

$$Circuits(I_A) = Graver(I_A)$$

Unimodular toric ideals

Theorem

For a unimodular toric ideal the set of circuits is the Graver basis of I_A.

Proof.

From the definition of unimodularity and the formula

$$\sum_{j=1}^{d+1}(-1)^j \det(a_{i_1}, a_{i_2}, \cdots, a_{i_{j-1}}, a_{i_{j+1}}, \cdots, a_{i_{d+1}}) e_{i_j},$$

every circuit has coordinates $1, -1$ or 0. Let $x^{u^+} - x^{u^-}$ be an element in the Graver basis of I_A. Then there exist a circuit $x^{c^+} - x^{c^-}$ such that $supp(c^+) \subset supp(u^+)$ and $supp(c^-) \subset supp(u^-)$. The monomials x^{c^+}, x^{c^-} are squarefree therefore $x^{c^+} | x^{u^+}$ and $x^{c^-} | x^{u^-}$. But $x^{u^+} - x^{u^-}$ is in the Graver basis and thus

$$x^{u^+} - x^{u^-} = x^{c^+} - x^{c^-}.$$

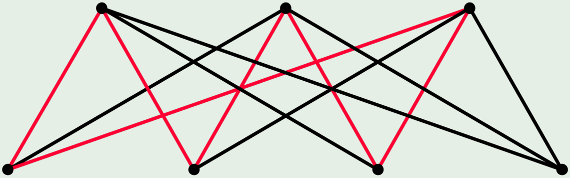
□

Unimodular toric ideals

Theorem

For a unimodular toric ideal I_A we have $Circuits_A = UGB_A = Graver_A$.

Example



A minimal Markov basis may differ from the above sets in a unimodular toric ideal.

Second Lawrence liftings

For $A \in \mathbb{M}^{m \times n}(\mathbb{Z})$ the second *Lawrence lifting* of A is denoted by $A^{(2)}$ and is the $(2m+n) \times 2n$ matrix

$$A^{(2)} = \begin{pmatrix} A & 0 \\ 0 & A \\ I_n & I_n \end{pmatrix} .$$

Note that $\mathrm{Ker}_{\mathbb{Z}}(A^{(2)}) = \{(u, -u) | u \in \mathrm{Ker}_{\mathbb{Z}}(A)\}$.
Therefore

$$I_{A^{(2)}} = < x^{u^+} y^{u^-} - x^{u^-} y^{u^+} | u \in \mathrm{Ker}_{\mathbb{Z}}(A) > \subset k[x_1, \dots, x_n, y_1, y_2, \dots, y_n].$$

Theorem

The binomial $x^{u^+} - x^{u^-}$ is in the Graver basis of I_A if and only if $x^{u^+} y^{u^-} - x^{u^-} y^{u^+}$ is in the Graver basis of $I_{A^{(2)}}$.

If $u = v +_c w$ then $(u, -u) = (v, -v) +_c (w, -w)$.

Second Lawrence Liftings

Theorem (Sturmfels)

For the toric ideal of the second Lawrence lifting the following sets of binomials coincide

1. *the Graver basis of* $I_{A^{(2)}}$
2. *the Universal Gröbner basis of* $I_{A^{(2)}}$
3. *any reduced Gröbner basis of* $I_{A^{(2)}}$
4. *any minimal Markov basis of* $I_{A^{(2)}}$.
5. *the set of indispensables of* $I_{A^{(2)}}$.

Proof.

Let M be a minimal Markov basis. We claim that M is the Graver basis. Suppose not, then there exists an element $x^{u^+}y^{u^-} - x^{u^-}y^{u^+}$ in the Graver basis of $I_{A^{(2)}}$ which can be written in the form

$$x^{u^+}y^{u^-} - x^{u^-}y^{u^+} =$$

$$f_1(x^{u_1^+}y^{u_1^-} - x^{u_1^-}y^{u_1^+}) + f_2(x^{u_2^+}y^{u_2^-} - x^{u_2^-}y^{u_2^+}) + \cdots + f_s(x^{u_s^+}y^{u_s^-} - x^{u_s^-}y^{u_s^+}),$$

for some polynomials $f_1, \cdots, f_s$ and some elements $x^{u_1^+}y^{u_1^-} - x^{u_1^-}y^{u_1^+}, x^{u_2^+}y^{u_2^-} - x^{u_2^-}y^{u_2^+}, \cdots, x^{u_s^+}y^{u_s^-} - x^{u_s^-}y^{u_s^+}$ of the Graver basis of $I_{A^{(2)}}$ different from $x^{u^+}y^{u^-} - x^{u^-}y^{u^+}$. This means that there exists an i such that $x^{u_i^+}y^{u_i^-}$ or $x^{u_i^-}y^{u_i^+}$ divides $x^{u^+}y^{u^-}$.
In the first case $x^{u_i^+}$ divides x^{u^+} and $y^{u_i^-}$ divides y^{u^-}, which implies that $x^{u_i^+}$ divides x^{u^+} and $x^{u_i^-}$ divides x^{u^-}. Thus $x^{u^+} - x^{u^-}$ does not belong to the Graver basis of I_A, and consequently $x^{u^+}y^{u^-} - x^{u^-}y^{u^+}$ is not in the Graver basis of $I_{A^{(2)}}$, a contradiction.
Similarly for the second case. □

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5. *the set of indispensables of* $I_{A^{(2)}}$.

Second Lawrence Liftings

This theorem gives a technique for computing the Graver basis of I_A.

Algorithm

Let $<$ be any monomial order on $k[x_1, \dots, x_n, y_1, y_2, \dots, y_n]$.

- *Compute the reduced Gröbner basis G of $I_{A^{(2)}}$.*
- *Substitute the y variables in the binomials of G by 1.*
- *The new binomials are in $k[x_1, \dots, x_n]$ and they are all the elements of the Graver basis of I_A.*

Theorem

The Graver basis of $I_{A^{(2)}}$ is the reduced Gröbner basis of $I_{A^{(2)}}$ with respect to any monomial order.

Theorem

The binomial $x^{u^+} - x^{u^-}$ is in the Graver basis of I_A if and only if $x^{u^+}y^{u^-} - x^{u^-}y^{u^+}$ is in the Graver basis of $I_{A^{(2)}}$.

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Theorem

The Graver basis of any toric ideal I_A is finite.

Toric ideals and Toric ideals of graphs

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This is a joint work with Enrique Reyes and Christos Tatakis

Graphs

Let G be the graph with vertices $V(G) = \{v_1, \dots, v_m\}$ and edges $E(G) = \{e_1, \dots, e_n\}$, where an edge $e \in E(G)$ is an unordered pair of vertices, $\{v_i, v_j\}$. Let A_G be the vertex-edge incident matrix of the graph G. The toric ideal I_G of the graph G is the toric ideal I_{A_G} in $\mathbb{K}[e_1, \dots, e_m]$, the polynomial ring in the m variables $e_1, \dots, e_m$ over a field $\mathbb{K}$.

Walks and subwalks

Definition

- A walk connecting $v_{i_1} \in V(G)$ and $v_{i_{q+1}} \in V(G)$ is a finite sequence of the form

$$w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$$

  with each $e_{i_j} = \{v_{i_j}, v_{i_{j+1}}\} \in E(G)$.
- We call a walk $w' = (e_{j_1}, \ldots, e_{j_t})$ a subwalk of w if $e_{j_1} \cdots e_{j_t} | e_{i_1} \cdots e_{i_q}$.
- A walk $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$ is called closed if $v_{i_{q+1}} = v_{i_1}$.
- A cycle is a closed walk

$$(\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_1}\})$$

  with $v_{i_k} \neq v_{i_j}$ for every $1 \leq k < j \leq q$.

Toric ideals of Graphs

Given an even closed walk

$$w = (e_{i_1}, e_{i_2}, \dots, e_{i_{2q}})$$

of the graph G we denote by

$$E^+(w) = \prod_{k=1}^{q} e_{i_{2k-1}}, \; E^-(w) = \prod_{k=1}^{q} e_{i_{2k}}$$

and by B_w the binomial

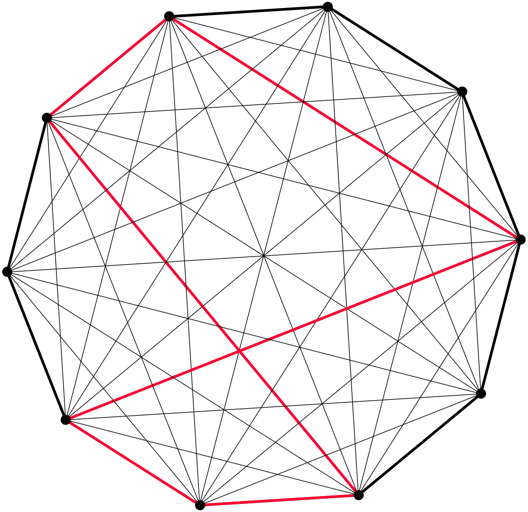
$$B_w = \prod_{k=1}^{q} e_{i_{2k-1}} - \prod_{k=1}^{q} e_{i_{2k}}$$

belonging to the toric ideal I_G.

Theorem (R. Villarreal)

The toric ideal I_G of a graph G is generated by binomials of the form B_w, where w is an even closed walk.

Toric ideals of Graphs



An even closed path in the complete graph with 10 vertices.

Graver basis

Definition

An irreducible binomial $x^{\mathrm{u}} - x^{\mathrm{v}}$ in I_A is called *primitive* if there exists no other binomial $x^{\mathrm{a}} - x^{\mathrm{b}} \in I_A$ such that x^{a} divides x^{u} and x^{b} divides x^{v}.

Definition

The set of all primitive binomials of a toric ideal I_A is called the Graver basis of I_A.

Toric ideals of Graphs

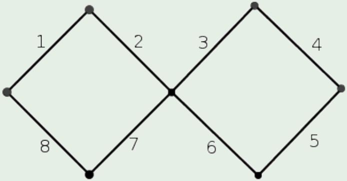
Definition

An even closed walk $w = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ is said to be primitive if $B_w \neq 0$ and there exists no even closed subwalk ξ of w of smaller length such that $E^+(\xi)|E^+(w)$ and $E^-(\xi)|E^-(w)$.

Theorem

The walk w is primitive if and only if the binomial B_w is primitive.

Example

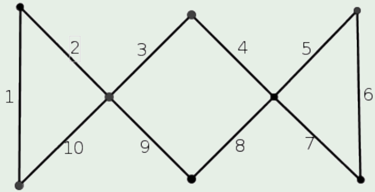


The walk $w = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)$ of the graph G is not primitive, since there exists a closed even subwalk of w, for example $\xi = (e_1, e_2, e_7, e_8)$ such that

$$e_1 e_7 | e_1 e_3 e_5 e_7 \text{ and } e_2 e_8 | e_2 e_4 e_6 e_8.$$

Note that $B_w = e_1 e_3 e_5 e_7 - e_2 e_4 e_6 e_8$ and $B_\xi = e_1 e_7 - e_2 e_8$.

Example

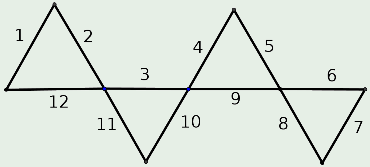


The walk $w = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10})$ in the graph G is primitive, although there exists an even closed subwalk $\xi = (e_3, e_4, e_8, e_9)$, but

- neither $e_3 e_8$ divides $e_1 e_3 e_5 e_7 e_9$
- nor $e_4 e_9$ divides $e_1 e_3 e_5 e_7 e_9$.

Note that $B_w = e_1 e_3 e_5 e_7 e_9 - e_2 e_4 e_6 e_8 e_{10}$ and $B_\xi = e_3 e_8 - e_4 e_9$.

Example



The walk $w = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12})$ is not primitive, since for the walk $\xi = (e_1, e_2, e_3, e_{10}, e_{11}, e_{12})$ we have that ξ is an even closed subwalk of w,

$$e_1 e_3 e_{11} | e_1 e_3 e_5 e_7 e_9 e_{11}$$

and

$$e_2 e_{10} e_{12} | e_2 e_4 e_6 e_8 e_{10} e_{12}.$$

Note that $B_w = e_1 e_3 e_5 e_7 e_9 e_{11} - e_2 e_4 e_6 e_8 e_{10} e_{12}$ and $B_\xi = e_1 e_3 e_{11} - e_2 e_{10} e_{12}$.

What are the primitive even closed walks?
A necessary characterization of the primitive elements was given by H. Ohsugi and T. Hibi :

Theorem

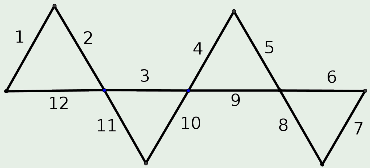
Let G be a finite connected graph. If $B \in I_G$ is primitive, then we have $B = B_w$ where w is one of the following even closed walks:

1. *w is an even cycle of G*
2. *$w = (c_1, c_2)$, where c_1 and c_2 are odd cycles of G having exactly one common vertex*
3. *$w = (c_1, w_1, c_2, w_2)$, where c_1 and c_2 are odd cycles of G having no common vertex and where w_1 and w_2 are walks of G both of which combine a vertex v_1 of c_1 and a vertex v_2 of c_2.*

It is easy to see that any binomial in the first two cases is always primitive but this is not true in the third case.

Non primitve walk

Example



Toric ideals of Graphs

In a toric ideal of a graph what are elements of the Graver basis?
What are the primitive even closed walks?

Theorem

Let G a graph and w an even closed walk of G. The walk w is primitive if and only if

1. *every block of w is a cycle or a cut edge,*
2. *every multiple edge of the walk w is a double edge of the walk and a cut edge of w,*
3. *every cut vertex of w belongs to exactly two blocks and it is a sink of both.*

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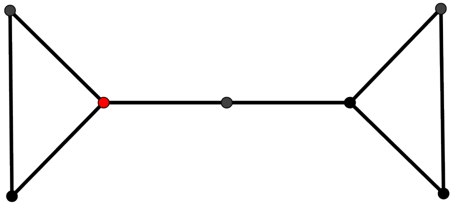
Toric ideals of Graphs

Definition

A cut vertex is a vertex of the graph whose removal increases the number of connected components of the remaining subgraph.



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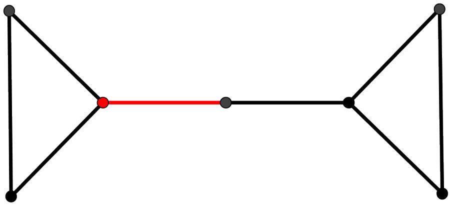
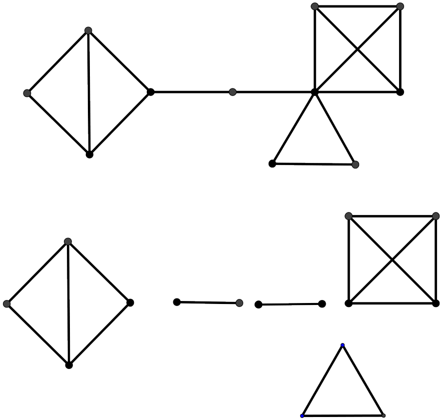


Definition

A graph is called biconnected if it is connected and does not contain a cut vertex.
A block is a maximal biconnected subgraph of a given graph G.



Toric ideals of Graphs

Definition

Every even primitive walk $w = (e_{i_1}, \ldots, e_{i_{2k}})$ partitions the set of edges in the two sets $w^+ = \{e_{i_j} | j \text{ odd}\}$, $w^- = \{e_{i_j} | j \text{ even}\}$ (otherwise the binomial B_w is not irreducible).
The edges of w^+ are called odd edges of the walk and those of w^- even edges.

Note that for a closed even walk whether an edge is even or odd depends only on the edge that you start counting from. So it is not important to identify whether an edge is even or odd but to separate the edges in the two disjoint classes.

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Definition

Every even primitive walk $w = (e_{i_1}, \dots, e_{i_{2k}})$ partitions the set of edges in the two sets $w^+ = \{e_{i_j} | j \text{ odd}\}, w^- = \{e_{i_j} | j \text{ even}\}$ (otherwise the binomial B_w is not irreducible).
The edges of w^+ are called odd edges of the walk and those of w^- even edges.

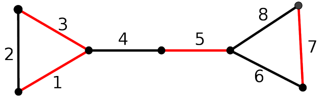
Note that for a closed even walk whether an edge is even or odd depends only on the edge that you start counting from. So it is not important to identify whether an edge is even or odd but to separate the edges in the two disjoint classes.

Toric ideals of Graphs

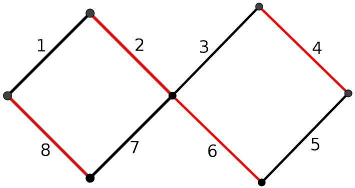
Definition

Sink of a block B is a common vertex of two odd or two even edges of the walk w which belong to the block B.

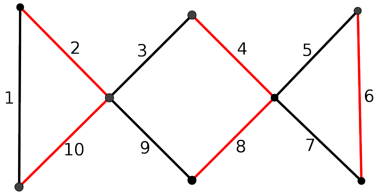
In particular if e is a cut edge of a primitive walk then e appears at least twice in the walk and belongs either to w^+ or w^-. Therefore both vertices of e are sinks.



Sink is a property of the walk w and not of the underlying graph w.



For example the walk $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)$ has no sink, while in the walk $(e_1, e_2, e_7, e_8, e_1, e_2, e_7, e_8)$ all four vertices are sinks.



The walk $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10})$ has two cut vertices which are both sinks of all of their blocks.

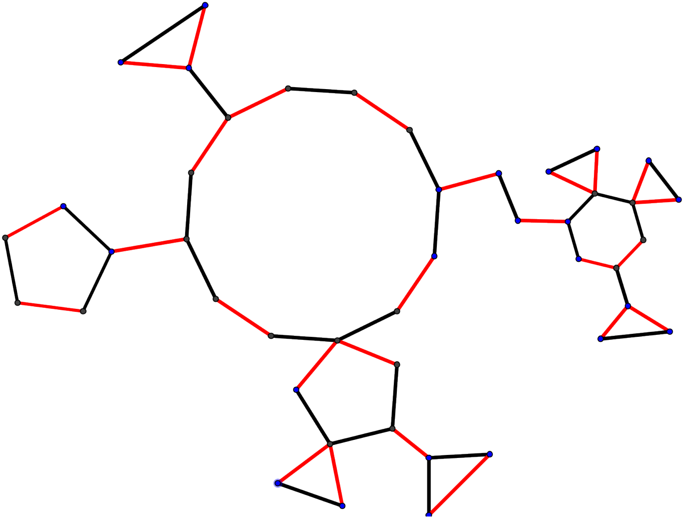
Toric ideals of Graphs

Theorem

Let G a graph and w an even closed walk of G. The walk w is primitive if and only if

1. *every block of* w *is a cycle or a cut edge,*
2. *every multiple edge of the walk w is a double edge of the walk and a cut edge of* w*,*
3. *every cut vertex of* w *belongs to exactly two blocks and it is a sink of both.*

Toric ideals of Graphs



Toric ideals and Toric ideals of graphs

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