Toric ideals of graphs

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This is a joint work with Enrique Reyes and Christos Tatakis

Toric ideals of graphs

- Graver basis
- universal Gröbner basis
- reduced Gröbner bases
- Circuits

- Graver basis
- minimal Markov bases
- indispensables

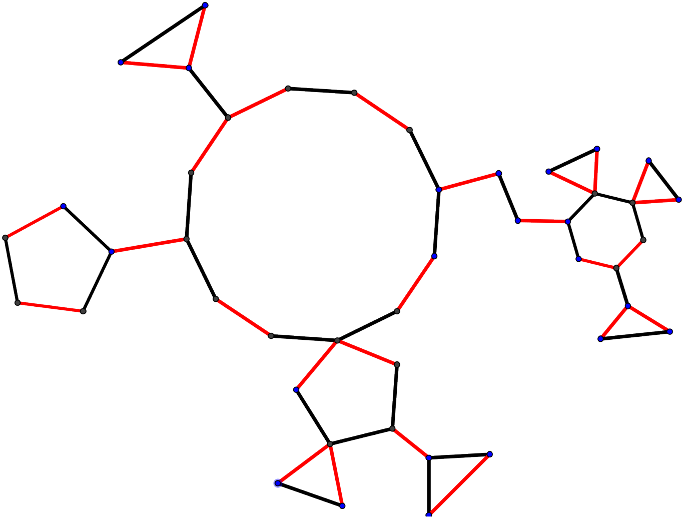
Graver basis

Theorem

Let G a graph and w an even closed walk of G. The walk w is primitive if and only if

1. *every block of* w *is a cycle or a cut edge,*
2. *every multiple edge of the walk w is a double edge of the walk and a cut edge of* w*,*
3. *every cut vertex of* w *belongs to exactly two blocks and it is a sink of both.*

Graver basis



Graver basis

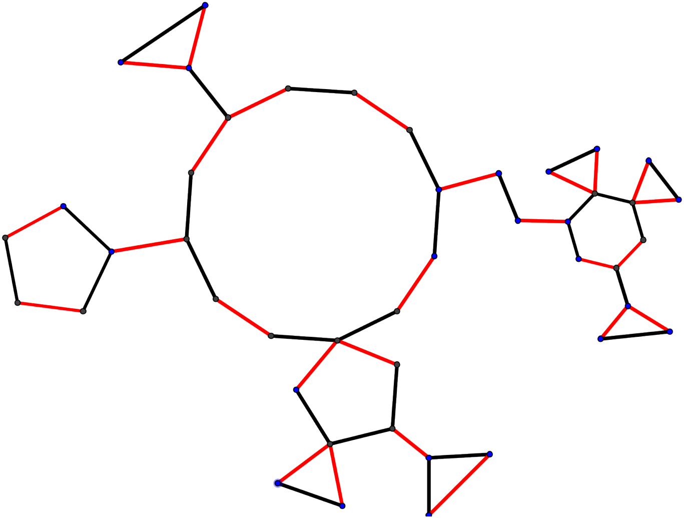
The following theorem describes the underlying graph of a primitive walk.

Theorem

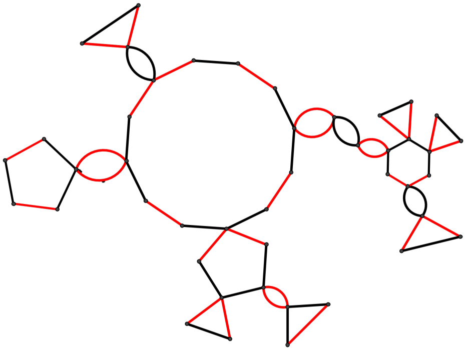
Let G be a graph and let W be a connected subgraph of G. The subgraph W is the graph w of a primitive walk w if and only if

1. *W is an even cycle or*
2. *W is not biconnected and*
    1. *every block of W is a cycle or a cut edge and*
    2. *every cut vertex of W belongs to exactly two blocks and separates the graph in two parts, the total number of edges of the cyclic blocks in each part is odd.*

Toric ideals of graphs



Toric ideals of Graphs



Replace every cut edge of W with two edges. Then the resulting graph W' is an Eulerian graph (it is connected and every vertex has degree even (two or four)). Any closed Eulerian trail of W' gives to an even closed walk of W which is primitive. Any other closed Eulerian trail gives rise to a different primitive walk but the corresponding binomials are the same or opposite.

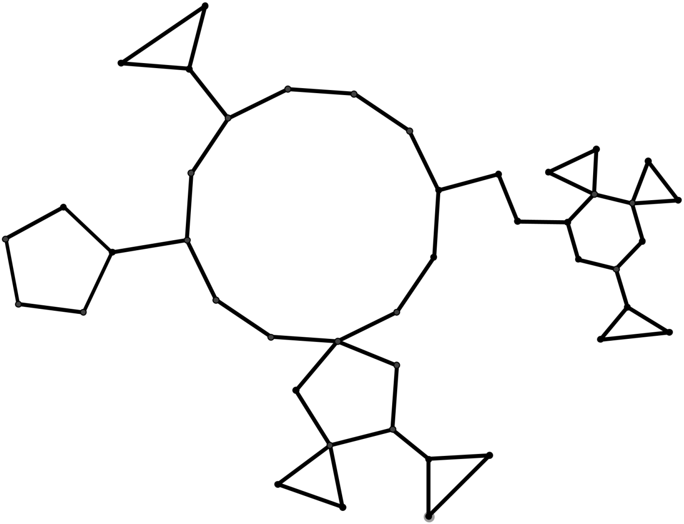
Circuits

An irreducible binomial belonging to I_A of minimal support is called a *circuit* of I_A.

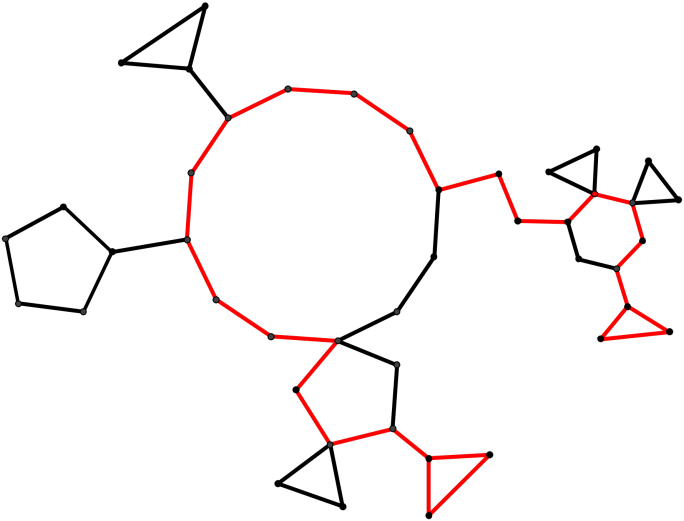
Theorem

(B. Sturmfels) The set of circuits of I_A is a subset of both the Universal Groebner basis and the Graver basis of I_A.

Circuits



Circuits



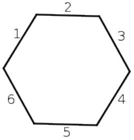
Circuits

A necessary and sufficient characterization of circuits was given by R. Villarreal:

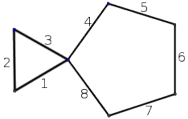
Theorem

Let G be a finite connected graph. The binomial $B \in I_G$ is circuit if and only if $B = B_w$ where

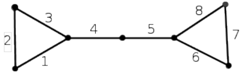
1. *w is an even cycle or*
2. *two odd cycles intersecting in exactly one vertex or*
3. *two vertex disjoint odd cycles joined by a path.*



w is an even cycle



two odd cycles intersecting in exactly one vertex



two vertex disjoint odd cycles joined by a path

The knowledge of the form of the circuits, the elements of the Graver basis, the minimal systems of generators and the elements of the universal Groebner basis of the toric ideal of a graph G, allow us to produce examples of toric ideals having specific properties.

True circuit conjecture

One of the fundamental problems in toric algebra is to give good upper bounds on the degrees of the elements of the Graver basis.

B. Sturmfels in 1995 with the help of S. Hosten and R. Thomas made the following conjecture:

Conjecture

The degree of any element in the Graver basis Gr_A of a toric ideal I_A is bounded above by the maximal true degree of any circuit in $\mathcal{C}_A$.

True circuit conjecture

Consider any circuit C of I_A and regard its support supp(C) as a subset of $A = \{a_1, \cdots, a_n\}$ (the set of the columns of A).

Definition

The index of the circuit C, index(C), is the index of $\mathbb{Z}$(supp(C)) in $\mathbb{R}$(supp(C))$\cap\mathbb{Z}A$.

Definition

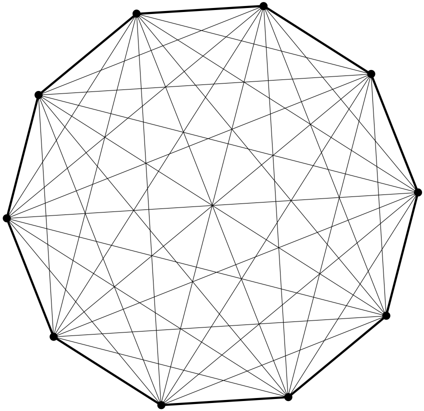
The *true degree* of the circuit C is the product $\deg(C)\cdot$index(C).

Theorem

The index(C) of a circuit C in the toric ideal of a graph is always 1. Therefore the true degree *of a circuit C in the toric ideal of a graph is the usual* $\deg(C)$.

True circuit conjecture

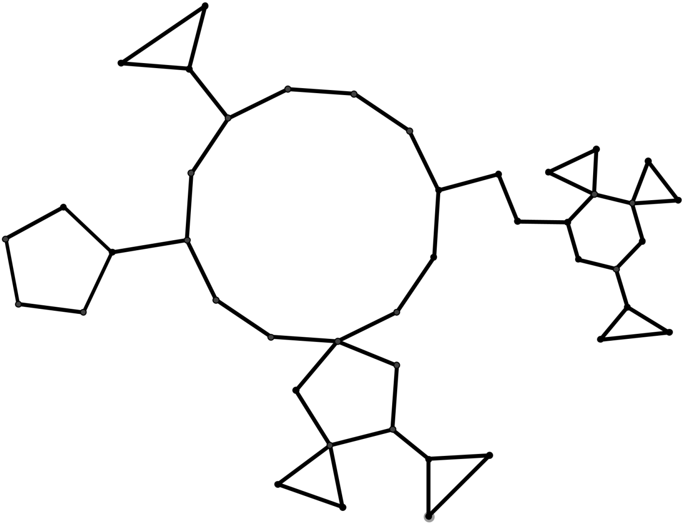
There are several examples of families of toric ideals where circuits do attain the maximum degree.
This is also true for families of toric ideals of graphs, for example the binomial that has the maximal degree in I_{K_n} is a circuit.



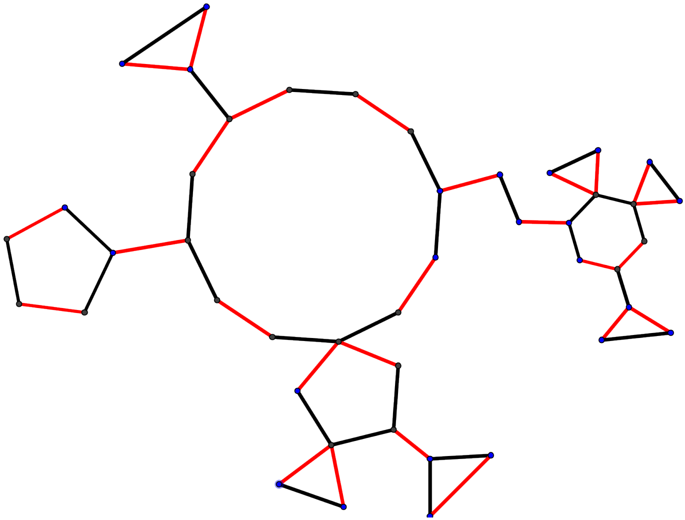
True circuit conjecture

But this is not true in the general case.

True circuit conjecture

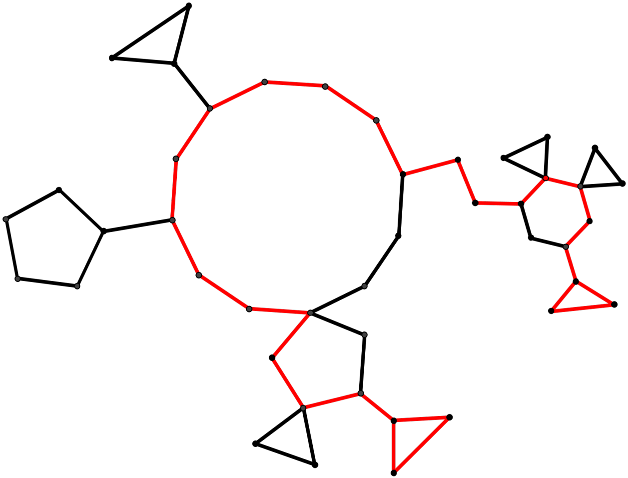


True circuit conjecture



Degree 30

True circuit conjecture



Graver degree $30 > 24$ maximal true circuit degree.

True circuit conjecture

Let I_A be a toric ideal. We denote by t_A the maximal true degree of a circuit in I_A.

The true circuit conjecture said:

$$deg(B) \leq t_A$$

for every element B in the Graver basis of I_A.

Question

Does the degree of any element in the Graver basis of a toric ideal I_A is bounded above by

- *a constant times t_A, say $10^{100} t_A$?*
- *or $10^{100}(t_A)^2$?*
- *or $10^{100}(t_A)^{2016}$?*

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- *or $10^{100}(t_A)^{2016}$?*

True circuit conjecture

Theorem

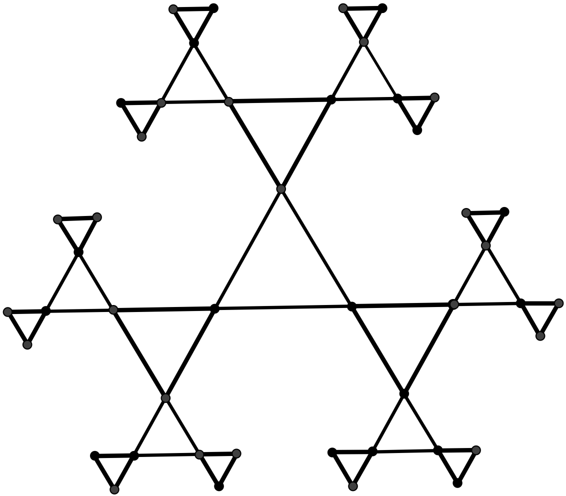
The degrees of the elements in the Graver basis of a toric ideal I_A cannot be bounded above by a polynomial in the maximal true degree of a circuit.

In particular there are examples of toric ideals I_A such that

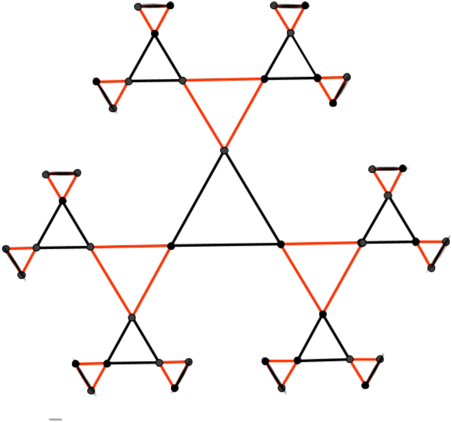
$$deg(B) > 10^{100}(t_A)^{2016}$$

for some B in the Graver basis of I_A.

True circuit conjecture

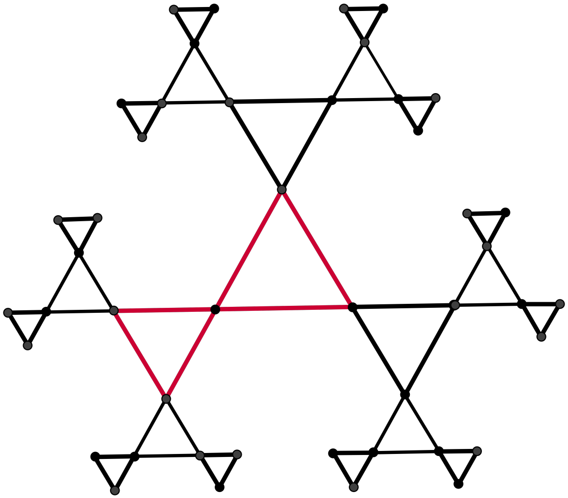


True circuit conjecture

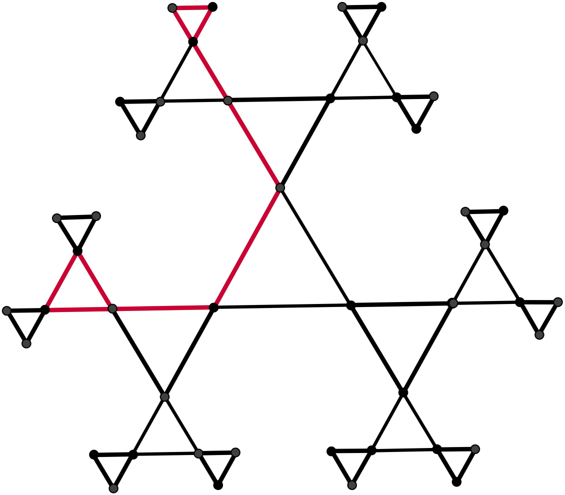


$$deg(B) = 9 \cdot 2^{r-1} - 3$$

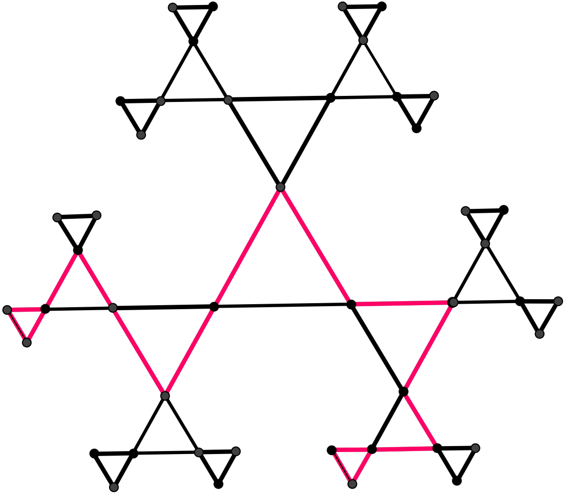
True circuit conjecture



True circuit conjecture

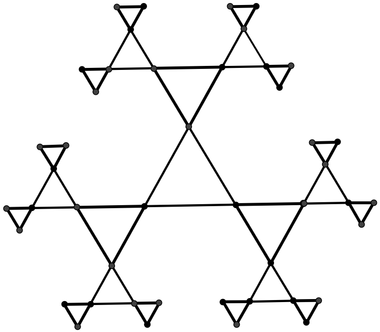


True circuit conjecture



$$t_A = 4r + 1$$

True circuit conjecture



$$deg(B) = 9 \cdot 2^{r-1} - 3$$

$$t_A = 4r + 1$$

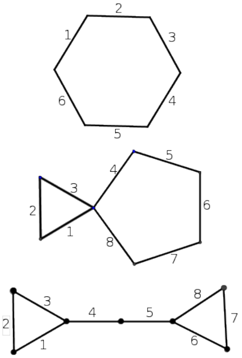
For large r we have

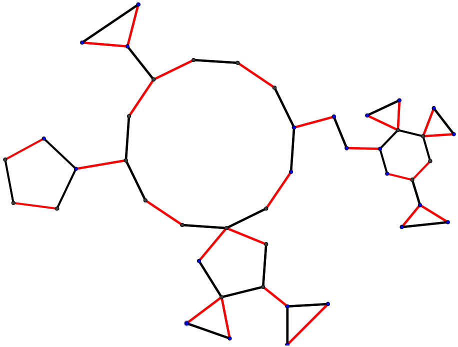
$$deg(B) = 9 \cdot 2^{r-1} - 3 > 10^{100}(4r+1)^{2016} = 10^{100}(t_A)^{2016}$$

Universal Gröbner bases

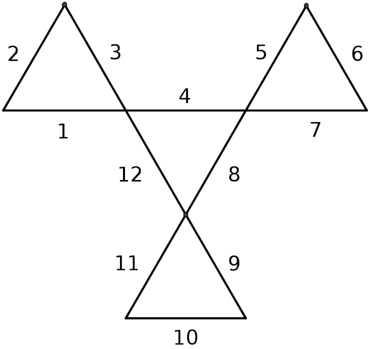
What is the Universal Gröbner basis of G for a general graph?

Universal Gröbner bases

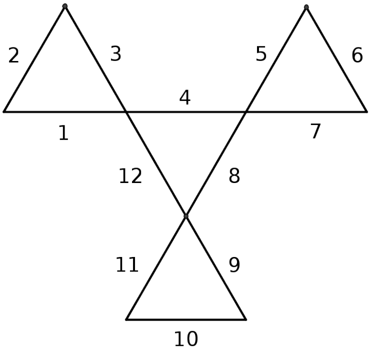




Universal Gröbner bases



Universal Gröbner bases



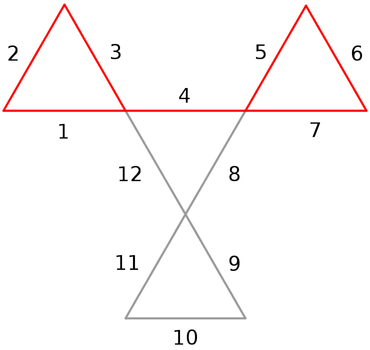
Let w be the walk
$(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12})$.
We claim that the binomial

$$B_w = e_1 e_3 e_5 e_7 e_9 e_{11} - e_2 e_4 e_6 e_8 e_{10} e_{12}$$

does not belong to the universal Gröbner basis of I_G.
Suppose that there exist a monomial order $<$ such that B_w belongs to the reduced Gröbner basis of I_G with respect to $<$.
There are two cases:

- $e_1 e_3 e_5 e_7 e_9 e_{11} > e_2 e_4 e_6 e_8 e_{10} e_{12}$
- $e_1 e_3 e_5 e_7 e_9 e_{11} < e_2 e_4 e_6 e_8 e_{10} e_{12}$

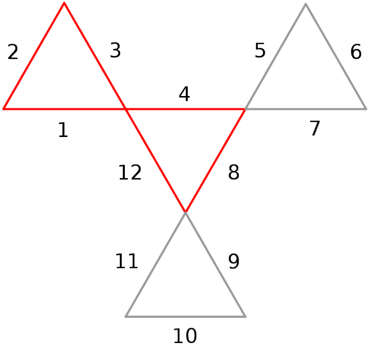
Universal Gröbner bases



First case:
$e_1e_3e_5e_7e_9e_{11} > e_2e_4e_6e_8e_{10}e_{12}$
Look at the binomials of I_G.
$B_1 = e_1e_3e_5e_7 - e_2e_4^2e_6$,
$B_2 = e_5e_7e_9e_{11} - e_6e_8^2e_{10}$,
$B_3 = e_9e_{11}e_1e_3 - e_{10}e_{12}^2e_2$.
Note that $e_1e_3e_5e_7 | e_1e_3e_5e_7e_9e_{11}$,
$e_5e_7e_9e_{11} | e_1e_3e_5e_7e_9e_{11}$, and
$e_9e_{11}e_1e_3 | e_1e_3e_5e_7e_9e_{11}$.
Therefore $e_1e_3e_5e_7 < e_2e_4^2e_6$,
$e_5e_7e_9e_{11} < e_6e_8^2e_{10}$,
$e_9e_{11}e_1e_3 < e_{10}e_{12}^2e_2$.

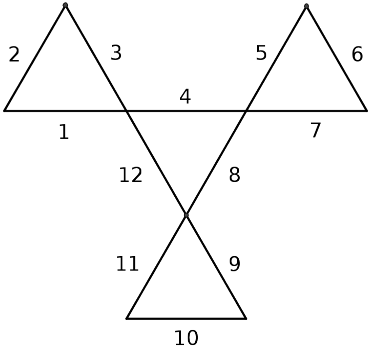
But then $(e_1e_3e_5e_7e_9e_{11})^2 < (e_2e_4e_6e_8e_{10}e_{12})^2$ contradicting $e_1e_3e_5e_7e_9e_{11} > e_2e_4e_6e_8e_{10}e_{12}$.

Universal Gröbner bases



Second case:
$e_1e_3e_5e_7e_9e_{11} < e_2e_4e_6e_8e_{10}e_{12}$
Look at the binomials of I_G.
$G_1 = e_1e_3e_8 - e_2e_4e_{12}$,
$G_2 = e_5e_7e_{12} - e_6e_8e_4$,
$G_3 = e_9e_{11}e_4 - e_{10}e_{12}e_8$.
Note that $e_2e_4e_{12} | e_2e_4e_6e_8e_{10}e_{12}$, $e_6e_8e_4 | e_2e_4e_6e_8e_{10}e_{12}$, and $e_{10}e_{12}e_8 | e_2e_4e_6e_8e_{10}e_{12}$.
Therefore $e_1e_3e_8 > e_2e_4e_{12}$, $e_5e_7e_{12} > e_6e_8e_4$, $e_9e_{11}e_4 > e_{10}e_{12}e_8$.

But then $(e_4e_8e_{12})(e_1e_3e_5e_7e_9e_{11}) > (e_4e_8e_{12})(e_2e_4e_6e_8e_{10}e_{12})$ contradicting $e_1e_3e_5e_7e_9e_{11} < e_2e_4e_6e_8e_{10}e_{12}$.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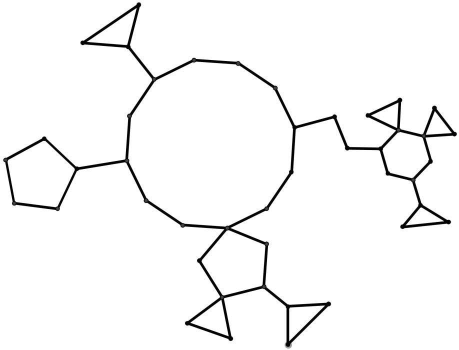

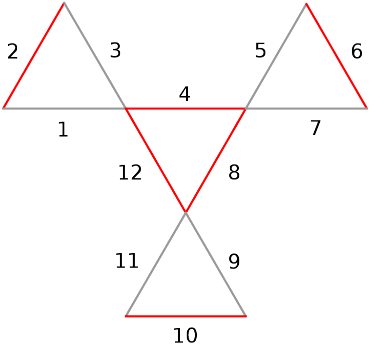
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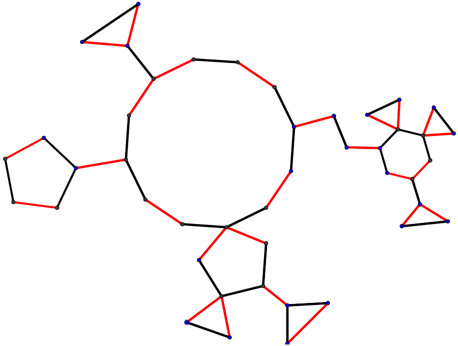
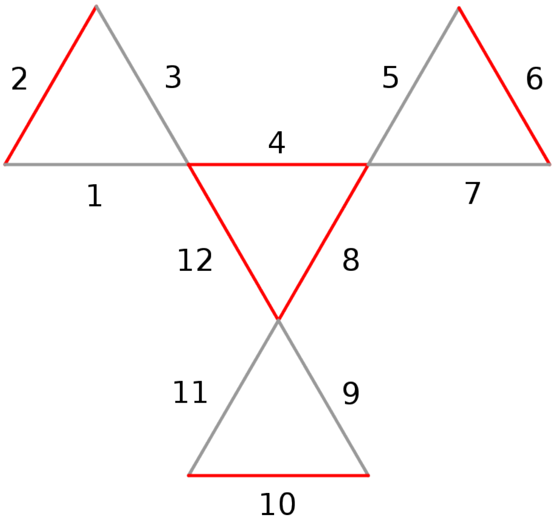


Universal Gröbner bases



$$B_w = e_1e_3e_5e_7e_9e_{11} - e_2e_4e_6e_8e_{10}e_{12}$$

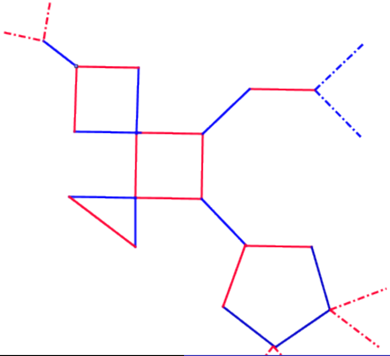
Universal Gröbner bases

Definition

A cyclic block *B* of a primitive walk *w* is called pure if all edges of *B* are either in w^+ or in w^-.

Theorem

Let w be an even primitive walk that has a pure cyclic block. Then B_w *does not belong to the universal Gröbner basis of* I_G*.*



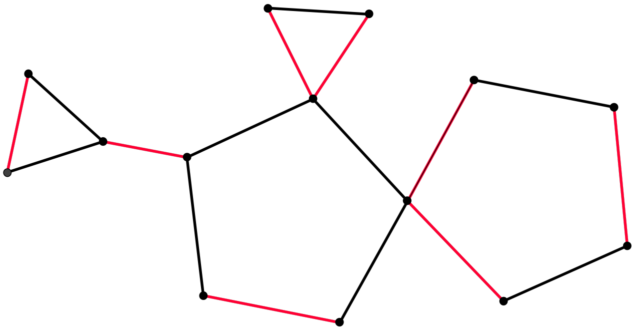
Universal Gröbner bases

Definition

A primitive walk w is called mixed if no cyclic block of w is pure.

Theorem

Let w be a primitive walk. B_w belongs to the universal Gröbner basis of I_G if and only if w is mixed.



Theorem

Let w be a primitive walk. B_w belongs to the universal Gröbner basis of I_G if and only if w is mixed.

Sketch of the proof. For any mixed primitive walk w we construct a term order $<_w$ that depends on w to prove that B_w belongs to the reduced Gröbner basis with respect to $<_w$.
It is enough to prove that whenever there exists a primitive binomial B_z such that $E^+(z)|E^+(w)$ then $E^-(z) >_w E^+(z)$. Note that $E^-(z) \nmid E^-(w)$ since w is primitive and $E^-(z) \nmid E^+(w)$ since w is mixed.

Universal Gröbner bases

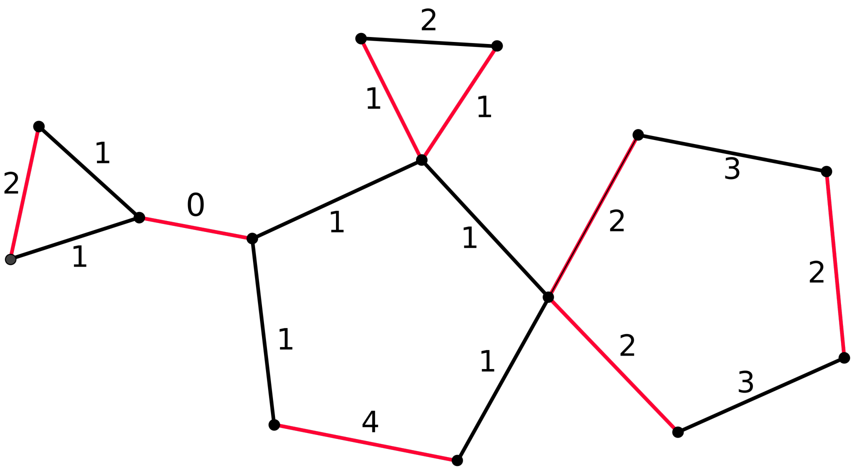
Let w be a mixed primitive walk. We define a term order $<_w$ on $\mathbb{K}[e_1, \ldots, e_n]$, as an elimination order with the variables that do not belong to w larger than the variables in w. We order the first set of variables by any term order, and the second set of variables as follows: Let $B_1, \ldots B_{s_0}$ be any enumeration of all cyclic blocks of w. Let t_i^+ denotes the number of edges in $\mathrm{w}^+ \cap B_i$ and t_i^- denotes the number of edges in $\mathrm{w}^- \cap B_i$. Let $W = (w_{ij})$ be the $(s_0) \times m$ matrix

$$w_{ij} = \begin{cases} 0, & \text{if } e_j \notin B_i, \\ t_i^-, & \text{if } e_j \in B_i \cap \mathrm{w}^+, \\ t_i^+, & \text{if } e_j \in B_i \cap \mathrm{w}^- \end{cases}$$

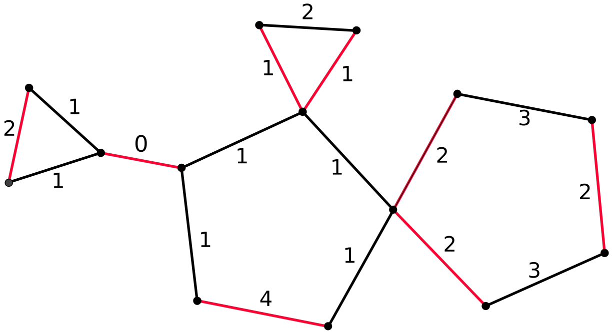
where m is the number of edges of w.
Let M be the matrix

$$\begin{pmatrix} I_{n\times n} & 0 \\ 0 & W \\ 0 & I_{m\times m} \end{pmatrix}.$$

Universal Gröbner bases



Universal Gröbner bases



$$\begin{pmatrix} I_{n\times n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 2 & 3 & 2 \\ 0 & & & & & & & & & I_{17\times 17} & & & & & & & & \end{pmatrix}.$$

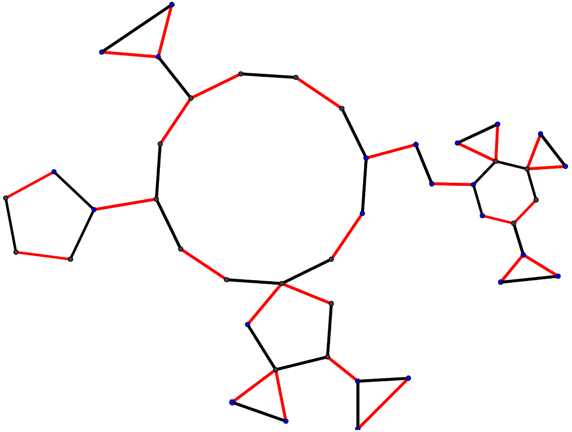
Universal Markov basis

The aim is to characterize the walks w of the graph G such that the binomial B_w belongs to a minimal Markov basis of the ideal I_G. Certainly the walk has to be primitive, but this is not enough. The walk must have more properties, the first one depends on the graph w and the rest on the induced graph G_w of w.

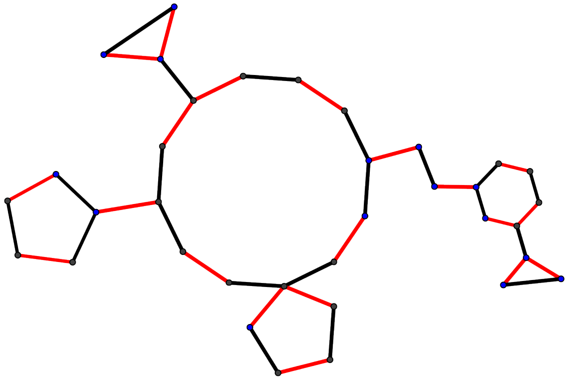
Universal Markov basis

Definition

We call strongly primitive walk a primitive walk that has not two cut points with distance one in any cyclic block.



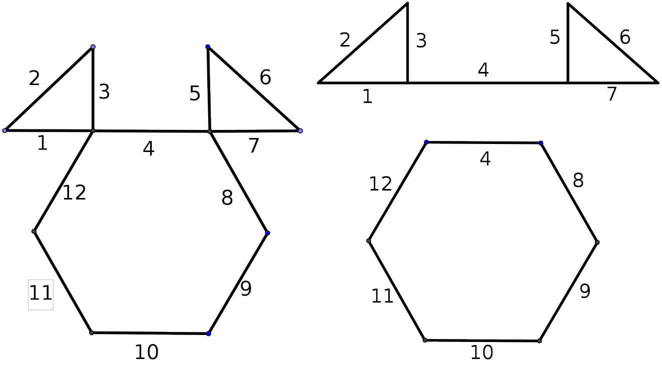
Universal Markov basis



Theorem

Let w be an even closed walk such that the binomial B_w is minimal then the walk w is strongly primitive.

Universal Markov basis



$$B_w = e_1e_3e_5e_7e_9e_{11} - e_2e_4e_6e_8e_{10}e_{12}$$

is not minimal since

$$B_w = (e_1e_3e_5e_7 - e_2e_4^2e_6)e_9e_{11} - (e_4e_9e_{11} - e_8e_{10}e_{12})e_2e_4e_6.$$

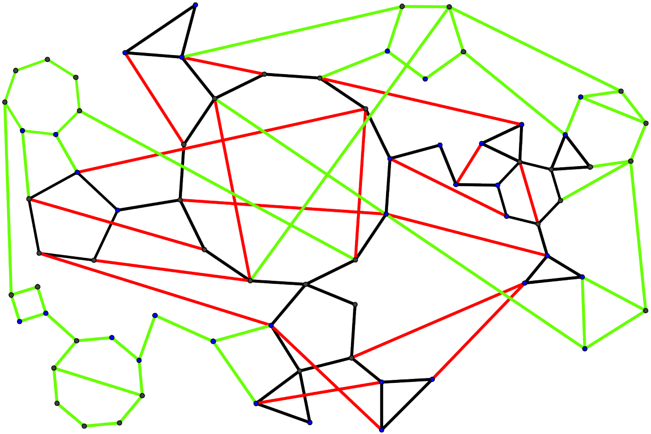
Universal Markov basis

While the property of a walk to be primitive depends only on the graph w, the property of the walk to be minimal or indispensable depends also on the induced graph G_w.

Definition

If W is a subset of the vertex set $V(G)$ of G then the *induced subgraph* of G on W is the subgraph of G whose vertex set is W and whose edge set is $\{\{v, u\} \in E(G) | v, u \in W\}$. When w is a closed walk we denote by G_w the induced graph of G on the set of vertices $V(\mathrm{w})$ of w.

Induced subraph



Universal Markov basis

An edge f of the graph G is called a chord of the walk w if the vertices of the edge f belong to $V(\mathrm{w})$ and $f \notin E(\mathrm{w})$.
In other words an edge is called chord of the walk w if it belongs to $E(G_w)$ but not in $E(\mathbf{w})$.
Let w be an even closed walk $((v_1, v_2), (v_2, v_3), \ldots, (v_{2k}, v_1))$ and $f = \{v_i, v_j\}$ a chord of w. Then f breaks w in two walks:

$$w_1 = (e_1, \ldots, e_{i-1}, f, e_j, \ldots, e_{2k})$$

and

$$w_2 = (e_i, \ldots, e_{j-1}, f),$$

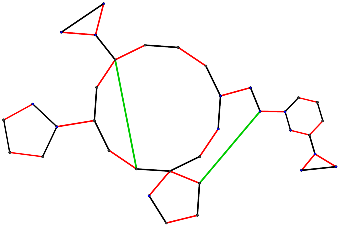
where $e_s = (v_s, v_{s+1}),\ 1 \leq s \leq 2k - 1$ and $e_{2k} = (v_{2k}, v_1)$. The two walks are both even or both odd.

Universal Markov basis

We partition the set of chords of a primitive even walk in three parts: bridges, even chords and odd chords.

Definition

A chord $f = \{v_1, v_2\}$ is called bridge of a primitive walk w if there exist two different blocks B_1, B_2 of w such that $v_1 \in B_1$ and $v_2 \in B_2$.



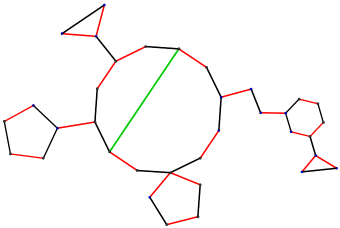
Theorem

Let w be a primitive walk. If B_w is a minimal binomial then w has no bridge.

Universal Markov basis

Definition

A chord is called even if it is not a bridge and breaks the walk in two even walks.
A chord is called odd if it is not a bridge and breaks the walk in two odd walks.

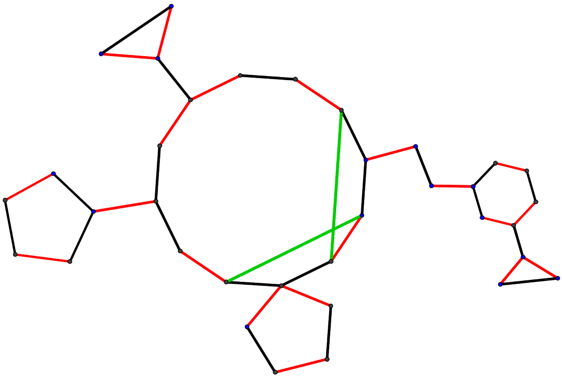


Theorem

Let w be a primitive walk. If B_w is a minimal binomial then w has no even chord.

Definition

Let $w = ((v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{2q}}, v_{i_1}))$ be a primitive walk. Let $f = \{v_{i_s}, v_{i_j}\}$ and $f' = \{v_{i_{s'}}, v_{i_{j'}}\}$ be two odd chords (that means not bridges and $j - s, j' - s'$ are even) with $1 \leq s < j \leq 2q$ and $1 \leq s' < j' \leq 2q$. We say that f and f' cross effectively in w if $s' - s$ is odd (then necessarily $j - s', j' - j, j' - s$ are odd) and either $s < s' < j < j'$ or $s' < s < j' < j$.

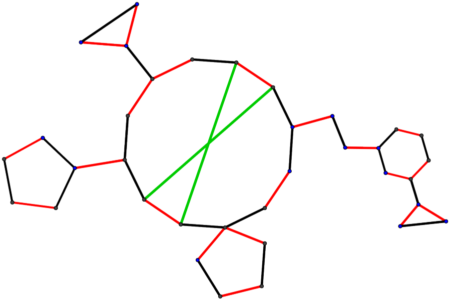


Universal Markov basis

Note that if two odd chords f and f' cross effectively in w then all of their vertices are in the same cyclic block of w.

Definition

We call an F_4 of the walk w a cycle (e, f, e', f') of length four which consists of two edges e, e' of the walk w either both odd or both even, and two odd chords f and f' which cross effectively in w.



Universal Markov basis

Definition

Let w be a primitive walk and f, f' be two odd chords. We say that f, f' cross strongly effectively in w if they cross effectively and they do not form an F_4 in w.

Theorem

Let w be a primitive walk. If B_w is a minimal binomial then all the chords of w are odd and there are not two of them which cross strongly effectively.

An F_4, (e_1, f_1, e_2, f_2), separates the vertices of w in two parts $V(\mathrm{w}_1)$, $V(\mathrm{w}_2)$, since both edges e_1, e_2 of the F_4 belong to the same block of $w = (w_1, e_1, w_2, e_2)$.

Definition

We say that an odd chord f of a primitive walk $w = (w_1, e_1, w_2, e_2)$ crosses an F_4, (e_1, f_1, e_2, f_2), if one of the vertices of f is in $V(\mathrm{w}_1)$, the other in $V(\mathrm{w}_2)$ and f is different from f_1, f_2.

Theorem

Let w be a primitive walk. If B_w is a minimal binomial, then no odd chord crosses an F_4 of the walk w.

Toric ideals of Graphs

Theorem

Let w be an even closed walk. B_w belongs to the universal Markov basis if and only if

1. *w is strongly primitive,*
2. *all the chords of w are odd and there are not two of them which cross strongly effectively and*
3. *no odd chord crosses an F_4 of the walk w.*

Indispensable binomials

Theorem

Let w be an even closed walk. B_w is an indispensable binomial if and only if w is a strongly primitive walk, all the chords of w are odd and there are not two of them which cross effectively.

We have that if B_w is indispensable then w has no F_4 and if B_w is minimal but not indispensable then w has at least one F_4. If no minimal generator has an F_4 then the toric ideal is generated by indispensable binomials, so the ideal I_G has a unique system of binomial generators and conversely.

Theorem

Let G be a graph which has no cycles of length four. The toric ideal I_G has a unique system of binomial generators.